# A Fourth-Order Accurate Multigrid Solver for Overset Grids

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We present a **fourth-order accurate** parallel multigrid solver for overset grids:

- The **overset grids of coarse levels** are automatically generated. The overlap grows as coarsing.
- The **coarse-grid operator** can be automatically generated by Galerkin averaging from the fine-grid operator.
- **Coarse-grid operators** of **second-order accuracy** is used to achieve convergence rates that are as good as those from using a fourth-order accurate coarse grid solver, with significantly more efficiency.
- **Numerical boundary conditions** are set to be consistent with the eigen-structure of the problem so that the convergence rate is not degraded.
- From a model problem, we obtain obtain optimal values of an **over-relaxation parameter** for the Red-Black smoother by *local Fourier analysis*.

## Elliptic boundary value problem on general geometry

$$\mathcal{L}u = f, \quad \mathbf{x} \in \Omega; \qquad \mathcal{B}u = g, \quad \mathbf{x} \in \Gamma.$$

- $\Omega \subset \mathbb{R}^d$, $d = 2,\ 3$.
- $L$ is a 2nd-order, linear, variable-coefficient, elliptic operator.
- $B$ is the boundary operator defining a Dirichlet, Neumann or mixed boundary condition.

**Discretization:**

$$\mathcal{L}_h u_h = f_h, \quad \mathbf{x_j} \in \Omega_h; \qquad \mathcal{B}_h u_h = g_h, \quad \mathbf{x_j} \in \Gamma_h; \qquad \mathcal{I}_h u_h = 0, \quad \mathbf{x_j} \in \Gamma_h^I.$$

- Construct an overlapping grid $\mathbf{G} = \{\mathbf{G}_g\}$, each component grid defined by a mapping from the unit square or cube to the physical space $\mathbf{C}_g : [0,\ 1]^d \to \mathbb{R}^d$.
- On each $\mathbf{G}_g$ the equations are transformed to the *unit square (or cube) coordinates* and discretized to 4th-order accuracy.
- Numerical boundary conditions are derived from $B$ as well as interpolation between component grids.

## Our algorithm

### The defect-correction multigrid algorithm for general $\mathcal{L}_h u_h = f_h$

$$u_h^n \xrightarrow{\mathcal{S}_h^{\nu_1}} \bar{u}_h^n \xrightarrow{f_h - \mathcal{L}_h \bar{u}_h^n} \bar{d}_h^n \qquad \hat{v}_h^n \xrightarrow{\bar{u}_h^n + \hat{v}_h^n} \bar{u}_h^{n+1} \xrightarrow{\mathcal{S}_h^{\nu_2}} u_h^{n+1}$$
$$\bar{d}_h^n \xrightarrow{\mathcal{I}_h^H} \bar{d}_H^n \xrightarrow{\tilde{\mathcal{L}}_H^{-1}} \bar{v}_H^n \xrightarrow{\mathcal{I}_H^h} \hat{v}_h^n$$

- **Coarse-grid correction** operator: $\mathcal{K}_h^H = \mathcal{I}_h - \mathcal{I}_H^h \tilde{\mathcal{L}}_H^{-1} \mathcal{I}_h^H \mathcal{L}_h$.
- **Iteration** operator (determining the convergence rate): $\mathcal{M}_h^H = \mathcal{S}_h^{\nu_2} \mathcal{K}_h^H \mathcal{S}_h^{\nu_1}$.
- Multigrid balances the reduction of low and high-frequency components of the error.

### Accelerated Red-Black smoothers

The **over-relaxation parameter** $\omega$ can be chosen so as to minimize

- The smoothing rate $\mu(\mathcal{S}_h(\omega);\ \nu_1,\ \nu_2) = \rho^{\frac{1}{\nu_1+\nu_2}}(\mathcal{S}_h^{\nu_2}\ \mathcal{Q}_h^H\ \mathcal{S}_h^{\nu_1})$ (where $\mathcal{Q}_h^H$ is the ideal coarse-grid correction operator that is identity on the high frequencies and 0 on the low frequencies);
- Instead, the overall multigrid convergence rate $\rho(\mathcal{M}_h^H)$. The idea is that we can choose the parameter optimizing the convergence rate for the **model problem** (*Poisson's equation on Cartesian grid*), in the anticipation that it will still be near optimal for a generalized problem that we need to solve.

## Coarse-grid operator $\mathcal{L}_H$

Given the **2nd-order** fine-grid operator $\mathcal{L}_h^{(2)}$:

- non-Galerkin: with the same difference stencil;
- **Galerkin:** $\mathcal{L}_H = \mathcal{I}_h^H\ \mathcal{L}_h^{(2)}\ \mathcal{I}_H^h$.

## What do the operators do on each frequency component?

- Galerkin coarse-grid correction operator $\mathcal{K}$ mimics the ideal one $\mathcal{Q}$ more closely.
- The main job of the coarse-grid correction operator is to deal with the frequency components near 0 where the smoother has the most trouble.

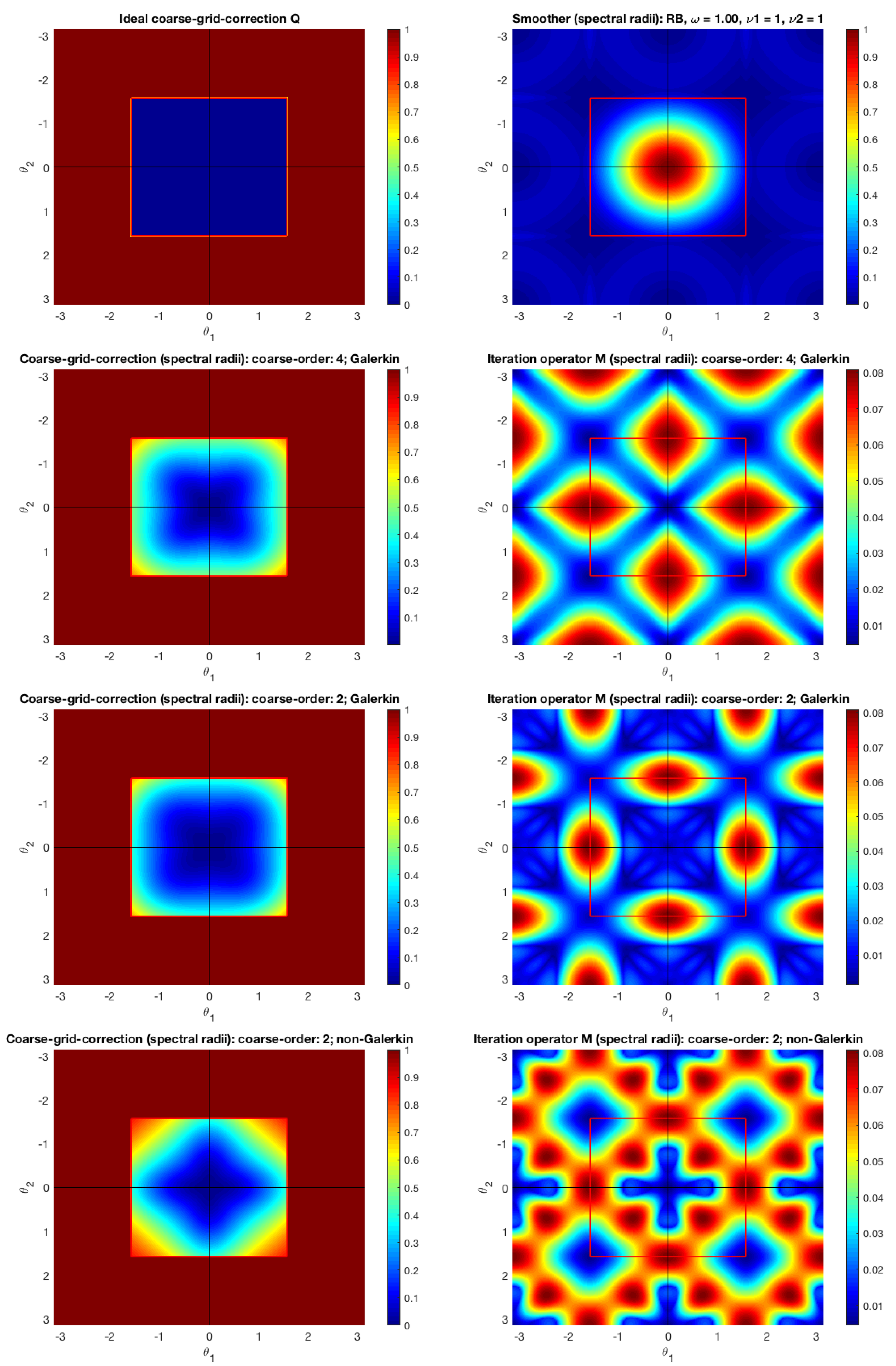


**Figure 1:** Operator illustrations of the 2-level, $V[1,\ 1]$-cycle for the 4th-order 2-D Poisson's equation on unit square with RB-GS smoother ($\omega = 1$) and **different coarse-grid operators**.

## Numerical results for general geometry are consistent with the model problem and local Fourier analysis

### Convergence

- **2nd-order Galerkin coarse-grid operator often yields the fastest convergence.**

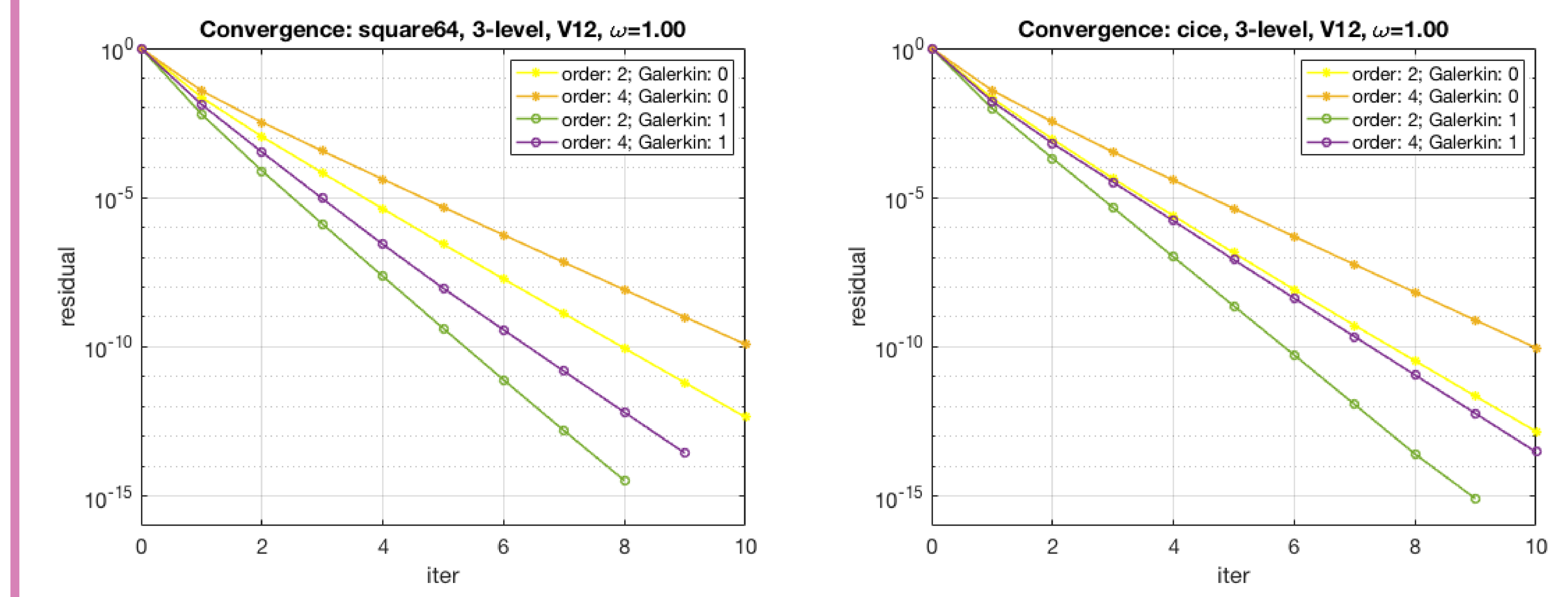


**Figure 2:** Convergence of the 3-level, $V[1,\ 2]$-cycle for the 4th-order 2-D Poisson's equation on different grids with RB-GS smoother ($\omega = 1$) and **different coarse-grid operators**. Left: square; right: Circle in a channel.

### Optimal $\omega$ for Red-Black smoother

- **2nd-order Galerkin coarse-grid operator often yields the fastest convergence.**
- **The optimal values of parameter $\omega$ from the model problem give optimal convergence rates for general geometry.**

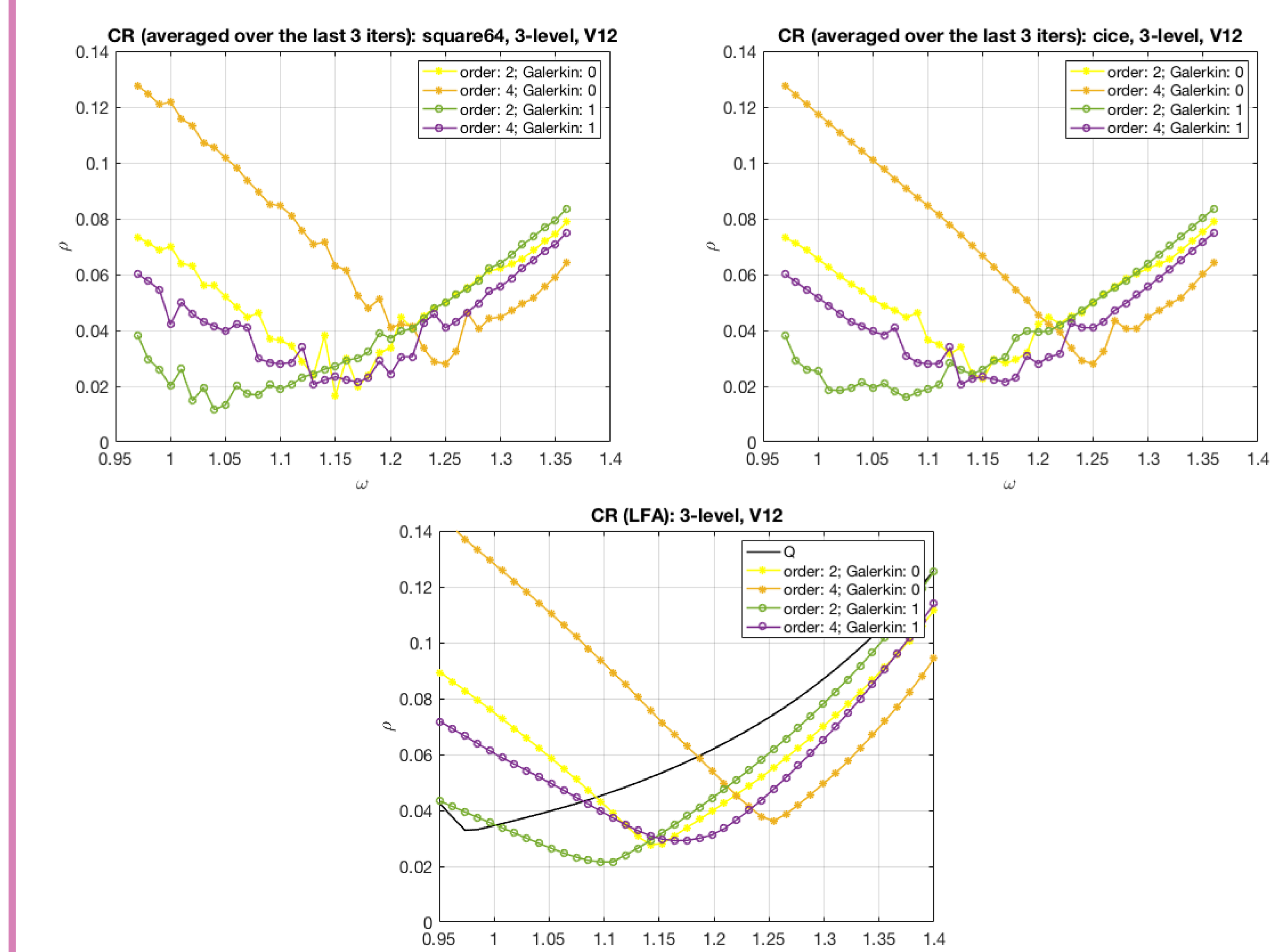


**Figure 3:** Convergence rates of the 3-level, $V[1,\ 2]$-cycle for the 4th-order 2-D Poisson's equation on unit square verses $\omega$ with RB-GS smoother and **different coarse-grid operators**. Top left: square; top right: Circle in a channel; bottom: local Fourier analysis.

## Conclusions

- Results from local Fourier analysis apply for general geometry.
- 2nd-order Galerkin coarse-grid operator often yields the fastest convergence.

## References

[1] William D. Henshaw. "On Multigrid For Overlapping Grids". In: *SIAM J. Sci. Comput.* 26.5 (2005), pp. 1547–1572.