Extending the Added-Mass Partitioned (AMP) Scheme for Solving FSI Problems Coupling Incompressible Flows with Elastic Beams to 3D

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Outline

- Introduction
- Governing Equations
- Added-Mass Partitioned (AMP) Interface Condition
- Euler-Bernoulli Beam Model
- Deforming Composite Grids
- Review of 2D results
- Working towards 3D
Introduction

Fluid-structure interaction (FSI) problems are important in many areas of engineering and applied science, such as modeling blood flow, aircraft, undersea cables and wind turbines, etc. In this talk, we focus on solving problems involving fluid interaction with elastic beams.

Numerical Methods

- **Monolithic Methods**
  - treat everything as a large system of evolution equations
  - advance the solutions together
  - less efficient
  - less flexible

- **Partitioned Methods**
  - reuse existing computational codes
  - successfully applied in many cases
  - stability issue arises for light beam
  - referred to as Added-Mass Instability
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Origin of Added-Mass Instabilities

in a vacuum

in a fluid
Origin of Added-Mass Instabilities

in a vacuum

in a fluid

Force
Origin of Added-Mass Instabilities

in a vacuum

in a fluid
Origin of Added-Mass Instabilities

in a vacuum

Solid simply moves according to Newton's laws of motion

in a fluid

Solid must displace fluid to move and therefore appears more massive than in vacuum, the so-called “added mass ($M_a$)”
Introduction

Partial Fixes of Added-Mass Instability

- Robin-Robin (mixed) boundary conditions with coefficients determined from simplified known solutions
- Interface artificial compressibility, fictitious pressure and fictitious mass
- Time-split interface conditions
- Semi-monolithic, approximate factorizations, Newton type schemes and fixed point iterations (Aitken accelerated)

References

Introduction

Traditional Scheme with Sub-Iterations (TS-SI)

- advance fluid (using interface velocity/position from the solid)
- advance solid (apply fluid forces to the solid)
- possibly iterate with under-relaxation to convergence for light solid (relaxed fixed-point method)

Diagram from Keyes et. al. 2012
Governing Equations

Fluid: \( x \in \Omega(t) \)

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho} \nabla \cdot \mathbf{\sigma}, \quad \nabla \cdot \mathbf{v} = 0,
\]

\[
\mathbf{\sigma} = -\rho \mathbf{I} + \mathbf{\tau}, \quad \mathbf{\tau} = \mu \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right],
\]

Beam: \( s \in \tilde{\Omega} \)

\[
\bar{\rho} \bar{A} \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{L}(\mathbf{u}, \mathbf{v}) + \mathbf{f}(s, t)
\]

\[
\tilde{\mathbf{f}}(s, t) = -\int_{\mathcal{P}} (\mathbf{n}) (\hat{\mathbf{\theta}}, s, t) \, d\hat{\mathbf{\theta}}
\]
**Governing Equations**

**Fluid:** $x \in \Omega(t)$

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \cdot \sigma + \frac{1}{\rho} \nabla \cdot \mathbf{v} = 0,
\]
\[
\sigma = -p\mathbf{I} + \tau, \quad \tau = \mu \left[ \nabla \mathbf{v} + \left( \nabla \mathbf{v} \right)^T \right],
\]

**Beam:** $s \in \bar{\Omega}$

\[
\bar{\rho}A \frac{\partial^2 \mathbf{\bar{u}}}{\partial t^2} = \bar{L}(\mathbf{\bar{u}}, \mathbf{\bar{v}}) + \mathbf{\bar{f}}(s, t)
\]
\[
\mathbf{\bar{f}}(s, t) = -\int_{\bar{P}} (\sigma \mathbf{n})(\dot{\theta}, s, t) d\dot{\theta}
\]

**Interface:** $\mathbf{v}(\bar{x}_b(\theta, s, t), t) = \bar{v}_b(\theta, s, t)$
AMP Scheme

We derive an Added-Mass Partitioned (AMP) scheme by matching the accelerations of fluid and beam at the interface $\Gamma(t)$:

$$\frac{D}{Dt} \mathbf{v}(\bar{x}_b(\theta, s, t), t) = \frac{\partial^2 \mathbf{u}}{\partial t^2} (s, t) + \frac{\partial}{\partial t} \mathbf{w}(\theta, s, t)$$

$$\Rightarrow \frac{\bar{\rho} \bar{A}}{\rho} \nabla \cdot \mathbf{\sigma}(\bar{x}_b(\theta, s, t), t) = \mathbf{L}(\bar{u}, \bar{v}) + \mathbf{f}(s, t) + \bar{\rho} \bar{A} \frac{\partial}{\partial t} \mathbf{w}(\theta, s, t)$$

Remark: $\mathbf{w}(\theta, s, t)$ is the finite-thickness correction of the beam velocity.

AMP Interface Condition

$$\int_{\mathcal{P}} (\mathbf{\sigma n})(\hat{\theta}, s, t) \, d\hat{\theta} + \frac{\bar{\rho} \bar{A}}{\rho} \nabla \cdot \mathbf{\sigma}(\bar{x}_b(\theta, s, t), t) = \mathbf{L}(\bar{u}, \bar{v}) + \bar{\rho} \bar{A} \frac{\partial}{\partial t} \mathbf{w}(\theta, s, t)$$
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\[
\Rightarrow \frac{\bar{\rho}A}{\rho} \nabla \cdot \sigma(\bar{x}_b(\theta, s, t), t) = \mathbf{L}(\bar{u}, \bar{v}) + \bar{f}(s, t) + \frac{\bar{\rho}A}{\partial t} \bar{w}(\theta, s, t)
\]

Remark: \( \bar{w}(\theta, s, t) \) is the finite-thickness correction of the beam velocity.

AMP Interface Condition

\[
\int_{\hat{\mathcal{P}}} (\sigma \mathbf{n})(\hat{\theta}, s, t) d\hat{\theta} + \frac{\bar{\rho}A}{\rho} \nabla \cdot \sigma(\bar{x}_b(\theta, s, t), t) = \mathbf{L}(\bar{u}, \bar{v}) + \frac{\bar{\rho}A}{\partial t} \bar{w}(\theta, s, t)
\]

Partitioned Schemes Using the AMP Condition

\[
\int_{\hat{\mathcal{P}}} (\sigma \mathbf{n})(\hat{\theta}, s, t) d\hat{\theta} + \frac{\bar{\rho}A}{\rho} \nabla \cdot \sigma(\bar{x}_b(\theta, s, t), t) = \mathbf{L}(\bar{u}^{(p)}, \bar{v}^{(p)}) + \frac{\bar{\rho}A}{\partial t} \bar{w}^{(p)}(\theta, s, t)
\]

where \( \bar{u}^{(p)} \), \( \bar{v}^{(p)} \) and \( \bar{w}^{(p)} \) are predicted solid variables.
AMP Scheme

The fluid equations are solved in the velocity-pressure formulation using a split-step scheme. [Henshaw & Petersson, 2001].

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho} \nabla \cdot \sigma, \quad \mathbf{x} \in \Omega
\]

\[
\Delta \rho = -\rho \nabla : (\nabla \mathbf{v})^T, \quad \mathbf{x} \in \Omega
\]

\[
\nabla \cdot \mathbf{v} = 0, \quad \mathbf{x} \in \partial \Omega
\]

**AMP Velocity Condition** (tangential component of AMP condition)

\[
t_m^T \left[ \int_{\bar{P}} (\tau\mathbf{n})(\hat{\theta}, \mathbf{s}, t) d\hat{\theta} + \frac{\mu \bar{\rho} \bar{A}}{\rho} \Delta \mathbf{v} \right] = t_m^T \left[ \frac{\bar{\rho} \bar{A}}{\rho} \nabla \rho + \bar{L}(\rho) + \bar{\rho} \bar{A} \frac{\partial}{\partial t} \bar{w}(\rho) + \int_{\bar{P}} (\rho\mathbf{n})(\hat{\theta}, \mathbf{s}, t) d\hat{\theta} \right]
\]

**AMP Pressure Condition** (normal component of AMP condition)

\[
\mathbf{n}^T \int_{\bar{P}} (\rho\mathbf{n})(\hat{\theta}, \mathbf{s}, t) d\hat{\theta} + \frac{\bar{\rho} \bar{A}}{\rho} \frac{\partial \rho}{\partial n} = \mathbf{n}^T \left[ -\bar{L}(\rho) - \bar{\rho} \bar{A} \frac{\partial \bar{w}(\rho)}{\partial t} + \frac{\mu \bar{\rho} \bar{A}}{\rho} \Delta \mathbf{v} + \int_{\bar{P}} (\tau\mathbf{n})(\hat{\theta}, \mathbf{s}, t) d\hat{\theta} \right]
\]
Euler-Bernoulli Beam model with $\mathbf{\tilde{u}} = [\tilde{u}_y, \tilde{u}_z]^T$ denoting the transverse displacements:

$$\rho A \frac{\partial^2 \mathbf{\tilde{u}}}{\partial t^2} = T \frac{\partial \mathbf{\tilde{u}}}{\partial s} - E J \frac{\partial^4 \mathbf{\tilde{u}}}{\partial s^4} + \mathbf{f},$$

The inertia matrix of the cross-section $A$ is

$$J = \int_A \begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix} dydz.$$

Note: $y$- and $z$-axes can be chosen such that is $J$ diagonal; i.e., the equations for $\tilde{u}_y$ and $\tilde{u}_z$ can be decoupled.
To reconstruct the surface:

\[
\begin{bmatrix}
  x(s, t) \\
  y(s, t) \\
  z(s, t)
\end{bmatrix} = R \begin{bmatrix}
  0 \\
  y_0(s) \\
  z_0(s)
\end{bmatrix} + \begin{bmatrix}
  s \\
  u_y(s, t) \\
  u_z(s, t)
\end{bmatrix} \quad \text{for } s \in [0, L],
\]

where \([x(s, t), y(s, t), z(s, t)])^T\) is coordinate of point \(P\) of the beam, \([s, y_0(s), z_0(s)])^T\) is its initial position, and \(R\) is a transformation matrix \((R = BT(\varphi))\), \(B\) is bending matrix and \(T\) is twisting matrix depending twist-angle \(\varphi\).
Deforming Composite Grids (DCG)

7. Numerical approach using deforming composite grids (DCG)

Our numerical approach for the solution of the equations governing an FSI initial-boundary-value problem is based on the use of deforming composite grids (DCG). This FSI-DCG approach was first described in [6] for the case of an inviscid compressible flow coupled to a linearly elastic solid, and later in [7] for the case of compressible flow coupled to nonlinear hyperelastic solids. Here, we extend the approach to FSI problems involving an incompressible flow coupled to deforming beams.

7.1. Deforming composite grids and the fluid domain solver

Deforming composite grids (DCGs) are used to discretize the evolving fluid domains in physical space. An overlapping grid, $G$, consists of a set of structured component grids, $\{G_g\}_{g=1}^{N}$, that cover each fluid domain, $\Omega(t)$, and overlap where the component grids meet. Typically, boundary-fitted curvilinear grids are used near the boundaries while one or more background Cartesian grids are used to handle the bulk of the fluid domain. Each component grid is a logically rectangular, curvilinear grid in $n_d$ space dimensions, and is defined by a smooth mapping from parameter space $r$ (the unit square or cube) to physical space $x$, $x = g(r,t)$, $r \in [0,1]^{n_d}$, $x \in \mathbb{R}^{n_d}$.

Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

Figure 5: Composite grids for a beam in a channel at times $t=0$, and $t=1$. The green fluid grid deforms over time to match the evolving beam (shown in white) and overlaps with the background Cartesian fluid grid (shown in blue). The reference curve of the beam is shown in red and the thickness of the beam describes the position of the surface of the beam.

In the FSI-DCG approach, component grids next to a fluid-structure interface deform over time to match the beam motion. This is illustrated in Figure 5 for the case of a beam in a fluid channel. A green fluid grid...
Deforming Composite Grids (DCG)

\[ Z \bar{h}^2 H \hat{p}(n+1) dy + Z H + \bar{h}^2 \hat{p}(n+1) + dy = 0, \]

which determines the one unknown \( \hat{v}(n+1) \). In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be

\[ \hat{v}(n+1) = \hat{v}_0 \pm 1, \]
\[ \hat{v}(n+1) = 0, \]
\[ \hat{\psi}_{n+1} = \hat{\psi}_0, \]
\[ \hat{p}(n+1) = \bar{K}_0 \hat{\psi}_0 D + D. \]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( k_x = 0 \) given in the continuous case by (50).

7. Numerical approach using deforming composite grids (DCG)

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\[ x = g(r, t), \]

\[ r \in [0, 1]^{n_d}, \]

\[ x \in \mathbb{R}^{n_d}. \]

Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

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\[ \sum_{\text{region } R} \left( \int_{\partial R} \left( \frac{\partial h}{\partial n} \hat{p}(n+1) + \hat{h} \right) + \int_{R} \hat{v}(n+1) \right) = 0, \]

which determines the one unknown \( \hat{v}(n+1) \).

In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be

\[ \hat{v}(n+1) = \hat{v}_0, \]
\[ \hat{v}(n+1) = 0, \]
\[ \hat{\alpha}_n + 1 = \hat{\alpha}_0, \]
\[ \hat{\alpha}(n+1) = \hat{\alpha}_0 D + \hat{\alpha}_0 D. \]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( k_x = 0 \) given in the continuous case by (50).

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Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

Figure 5: Composite grids for a beam in a channel at times \( t = 0, 0.5, 1.0 \).

The green fluid grid deforms over time to match the evolving beam (shown in white) and overlaps with the background Cartesian fluid grid (shown in blue). The reference curve of the beam is shown in red and the thickness of the beam describes the position of the surface of the beam.

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Deforming Composite Grids (DCG)

Composite grids consist of a set of structured component grids that cover the domain and overlap where the component grids meet.

Each component grid is a logically rectangular curvilinear grid.

Solutions on different component grids are coupled by interpolation.

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for $kx = 0$ given in the continuous case by (50).

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Deforming Composite Grids (DCG)

The single constraint
\[ Z \bar{h}^2 H \hat{p}(n+1) dy + Z H + \bar{h}^2 \hat{p}(n+1) + dy = 0, \]
which determines the one unknown \( \hat{\zeta} \). In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be
\[
\hat{v}(n+1) = \hat{v}_0, \\
\hat{v}(n+1) = 0, \\
\hat{\zeta}_{n+1} = \hat{\zeta}_0, \\
\hat{p}(n+1) = \hat{p}_0 D \hat{\zeta}_0 D + D.
\]

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Deforming Composite Grids (DCG)

\[ \int Z \bar{h}^2 H \hat{p}(n+1) \, dy + \int H + \bar{h}^2 \hat{p}(n+1) \, dy = 0, \]

which determines the one unknown \( \hat{v}(n+1) \). In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be

\[ \hat{v}(n+1) = v_0, \quad \hat{v}(n+1) = 0, \quad \hat{\phi}(n+1) = \hat{\phi}_0, \quad \hat{p}(n+1) = \hat{p}_0 D. \]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( k_x = 0 \) given in the continuous case by (50).

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\[ x = g(r, t), \quad r \in [0, 1]^{n_d}, \quad x \in \mathbb{R}^{n_d}. \]

Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

- Composite grids consist of a set of structured component grids that cover the domain and overlap where the component grids meet.
- Each component grid is a logically rectangular curvilinear grid.
- Solutions on different component grids are coupled by interpolation.
- The green fluid grid deforms over time to match the evolving beam (shown in white) and overlaps with the background Cartesian fluid grid (shown in blue).
- The reference curve of the beam is shown in red.
Deforming Composite Grids (DCG)

\[
Z \bar{h}^2 H \hat{p}(n+1) \, dy + Z H + \bar{h}^2 \hat{p}(n+1) \, dy = 0,
\]

which determines the one unknown \( \hat{\nu}(n+1) \). In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be

\[
\hat{\nu}(n+1) = \hat{\nu}_0 \pm 1, \\
\hat{\nu}(n+1) = 0, \\
\hat{\phi}^{n+1} = \hat{\phi}^0, \\
\hat{p}(n+1) = \hat{\phi}_0 \bar{K}_0 \hat{\phi}^0 D \hat{\phi} D + D.
\]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( k_x = 0 \) given in the continuous case by (50).

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Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

![Composite grids for a beam in a channel at times t=0, and 1.0. The green fluid grid deforms over time to match the evolving beam (shown in white) and overlaps with the background Cartesian fluid grid (shown in blue). The reference curve of the beam is shown in red and the thickness of the beam describes the position of the surface of the beam.](image)

In the FSI-DCG approach, component grids next to a fluid-structure interface deform over time to match the beam motion. This is illustrated in Figure 5 for the case of a beam in a fluid channel. A green fluid grid...
Deforming Composite Grids (DCG)

\[
Z \bar{h}^2 H \hat{p}(n+1) dy + Z H + \bar{h}^2 \hat{p}(n+1) + dy = 0,
\]
which determines the one unknown \( \beta^{(n+1)} \).

In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( k_x = 0 \), and the solution is found to be

\[
\hat{v}^{(n+1)} = \hat{v}_0^{(n+1)} \pm 1,
\hat{v}^{(n+1)} = 0,
\hat{\beta}^{n+1} = \hat{\beta}_0^{n+1},
\hat{p}^{(n+1)} = \alpha \bar{K}_0 \hat{\beta}_0^D + D \ alph\text{\texttt{\texttt{D}}}.\]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( k_x = 0 \) given in the continuous case by (50).

7. Numerical approach using deforming composite grids (DCG)

Our numerical approach for the solution of the equations governing an FSI initial-boundary-value problem is based on the use of deforming composite grids (DCG). This FSI-DCG approach was first described in [6] for the case of an inviscid compressible flow coupled to a linearly elastic solid, and later in [7] for the case of compressible flow coupled to nonlinear hyperelastic solids. Here, we extend the approach to FSI problems involving an incompressible flow coupled to deforming beams.

7.1. Deforming composite grids and the fluid domain solver

Deforming composite grids (DCGs) are used to discretize the evolving fluid domains in physical space. An overlapping grid, \( G \), consists of a set of structured component grids, \( \{G_g\} \), \( g = 1, \ldots, N \), that cover each fluid domain, \( \Omega_k(t) \), and overlap where the component grids meet. Typically, boundary-fitted curvilinear grids are used near the boundaries while one or more background Cartesian grids are used to handle the bulk of the fluid domain. Each component grid is a logically rectangular, curvilinear grid in \( n_{\text{d}} \) space dimensions, and is defined by a smooth mapping from parameter space \( \mathbf{r} \) (the unit square or cube) to physical space \( \mathbf{x} \), \( \mathbf{x} = g(\mathbf{r}, t) \), \( \mathbf{r} \subset [0, 1]^{n_{\text{d}}} \), \( \mathbf{x} \subset \mathbb{R}^{n_{\text{d}}} \).

Typically, the background grids are static, while the boundary-fitted grids evolve in time to match the motion of the boundary.

Figure 5: Composite grids for a beam in a channel at times \( t = 0, 0.5, 1 \).

The green fluid grid deforms over time to match the evolving beam (shown in white) and overlaps with the background Cartesian fluid grid (shown in blue). The reference curve of the beam is shown in red and the thickness of the beam describes the position of the surface of the beam.

In the FSI-DCG approach, component grids next to a fluid-structure interface deform over time to match the beam motion. This is illustrated in Figure 5 for the case of a beam in a fluid channel. A green fluid grid

Fluid equations are discretized using Curvilinear Finite-Difference method in space
Deforming Composite Grids (DCG)

\[ \int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega + \int_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \, dS = 0, \]

which determines the one unknown \( t \). In this case it can be shown that the AMP scheme is stable with no time-step restriction for \( kx = 0 \), and the solution is found to be

\[ \hat{v}(n+1) \pm 1 = \hat{v}_0 \pm 1, \]
\[ \hat{v}(n+1) \pm 2 = 0, \]
\[ \hat{\imath}n+1 = \hat{\imath}0, \]
\[ \hat{p}(n+1) \pm \bar{h}K0 \hat{\imath}0 D \hat{D} + D \hat{p} = \text{const}. \]

Note that, by adding one pressure regularization, the AMP scheme is able to determine the solution uniquely, and the solution is compatible with the exact solution for \( kx = 0 \) given in the continuous case by (50).

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\[ x = g(r, t), \quad r \in [0, 1]^n_d, \quad x \in \mathbb{R}^{n_d}. \]

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Deforming Composite Grids (DCG)
Review of 2D results

Stability Analysis
- Small beam deformation
- Fixed fluid domains
- Periodic boundary conditions in the $x$.

Theorem

When $k \neq 0$, the AMP Algorithm is stable if and only if

$$\Delta t < 2 \sqrt{\frac{\bar{\rho} h + \hat{M}_a^+ (k) + \hat{M}_a^- (k)}{\bar{L}}}.$$  

When $k = 0$, the scheme is non-dissipative.

Theorem

Assuming $\bar{\rho} h > 0$ and $\bar{L} > 0$, the TP Algorithm is weakly stable if and only if

$$\Delta t < 2 \sqrt{\frac{\bar{\rho} h - \hat{M}_a^- (k) - \hat{M}_a^+ (k)}{\bar{L}}}.$$  

Here $\hat{M}_a^\pm (k)$ is represents the added-masses in the upper and lower fluid chambers.
Review of 2D results

Manufactured Solutions

\[ u_e(x, y, t) = -a \cos(f_x \pi x) \sin(f_x \pi (y - \eta_e - 1)) \cos(f_t \pi t), \]

\[ v_e(x, y, t) = a \sin(f_x \pi x) \cos(f_x \pi (y - \eta_e - 1)) \cos(f_t \pi t) - a \cos(f_x \pi x) \sin(f_x \pi (y - \eta)) \frac{\partial \eta_e}{\partial x} \cos(f_t \pi t), \]

\[ p_e(x, y, t) = \cos(f_x \pi x) \cos(f_x \pi y) \cos(f_t \pi t), \]

\[ \eta_e(x, t) = \frac{a}{\pi f_t} \sin(f_x \pi x) \sin(f_t \pi t). \]

- The exact solution is divergence free.
- Fluid and Beam velocities match at the interface.

Computational results

Grid at \( t = 0.1 \)

\( v_2 \) at \( t = 0.1 \)

Error in \( v_2 \) at \( t = 0.1 \)

light beam (\( \bar{\rho} \bar{h} = 10^{-3} \))
Review of 2D results

Convergence Rate

light beam ($\bar{\rho}\bar{h} = 1.0e - 03$)

- No sub-iterations needed for light beam
- Second order accurate for all solution components
- Second order accurate for all beam densities (light, medium and heavy)

medium beam ($\bar{\rho}\bar{h} = 1.0e + 00$)

heavy beam ($\bar{\rho}\bar{h} = 1.0e + 03$)
Review of 2D results

Run-time Performance Comparison: AMP vs. TP-SI

<table>
<thead>
<tr>
<th>$\bar{\rho} = 1$</th>
<th>AMP s/step</th>
<th>TP-SI s/step</th>
<th>sub-its</th>
<th>AMP speed-up TP-SI/AMP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\rho} = 1$</td>
<td>2.87</td>
<td>22.6</td>
<td>62</td>
<td>7.87</td>
</tr>
<tr>
<td>$\bar{\rho} = 100$</td>
<td>1.33</td>
<td>1.48</td>
<td>0</td>
<td>1.1</td>
</tr>
</tbody>
</table>

- Flow past a light and heavy beam in a channel computed using grid $G^{(8)}$ (150K grid points)
- The majority of CPU time is spent in solving the pressure equation
- Pressure is solved with a bi-Conjugate-Gradient-stabilized Krylov solver and an ILU(3) preconditioner with a relative convergence tolerance of $10^{-6}$
Review of 2D results

Four Beams in A channel

Computed using grid $G^{(16)}$ (760K grid points).

Beam tip motions

- **Beam 1** ($\bar{\rho}\bar{A} = 2.0e + 02$, $\bar{E}\bar{I} = 30.0$)
- **Beam 2** ($\bar{\rho}\bar{A} = 2.0e + 00$, $\bar{E}\bar{I} = 10.0$)
- **Beam 3** ($\bar{\rho}\bar{A} = 2.0e - 01$, $\bar{E}\bar{I} = 20.0$)
- **Beam 4** ($\bar{\rho}\bar{A} = 2.0e + 01$, $\bar{E}\bar{I} = 30.0$)

Figure 4bic_AMP_NB_G16_Crop.mp4
Numerical Challenges Extending to 3D

- To implement the AMP condition, we need to evaluate

\[ n^T \int_{\bar{P}} (\rho n)(\hat{\theta}, s, t) \, d\hat{\theta} \quad \text{and} \quad n^T \int_{\bar{P}} (\tau n)(\hat{\theta}, s, t) \, d\hat{\theta} \]

- Need efficient and accurate quadrature to evaluate the line integrals on the beam surface grid (overset grids for more complex beams).
- Grid generation for deforming beams
- Parallelize for large scale problems
Working towards 3D

Preliminary Results in 3D

fig/beamins3d.mp4

Prescribed Motion
Working towards 3D

Preliminary Results in 3D

fig/beamins3d.mp4

Traditional Scheme
Thank you!