An Overset Mesh Framework for the Hybridizable Discontinuous Galerkin Method

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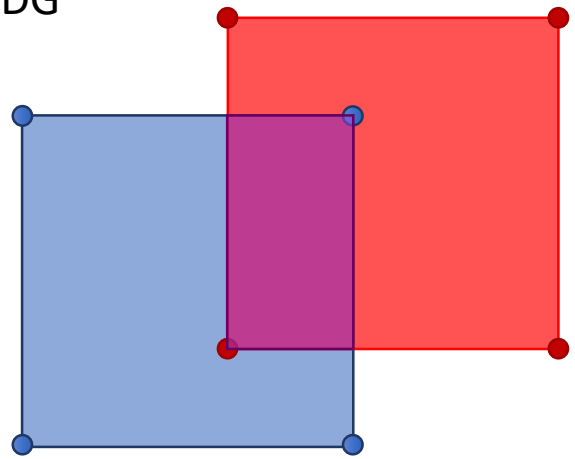
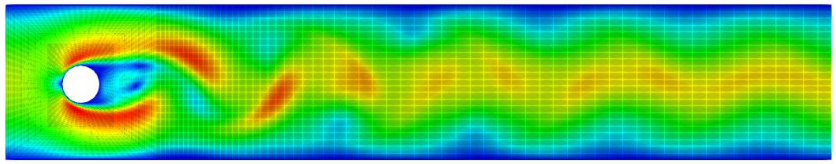
Presentation Outline

- Motivation
- Introduction
  - Overset meshing
  - Hybridizable discontinous Galerkin (HDG) discretization

- Overset-HDG



- Computational Studies
  - Elasticity
  - Navier-Stokes



  - Arbitrary Lagrangian-Eulerian (ALE) Navier-Stokes

- Future Work
  - Dynamic ALE
  - Fluid-Structure Interaction (FSI)
  - Robustness, scaling, and efficiency
  - Incorporation of hole cutting

Motivation

Develop a robust and efficient computational tool that can accurately simulate tightly coupled nonlinear physics in a single solver, specifically large deformation FSI.

Goals

- Extend overset discontinuous Galerkin[1,2] (DG) schemes to HDG
- Extend HDG FSI[3] into the overset mesh framework

Scope

- Derive an overset-HDG method and show no loss off accuracy
- Construct the essential communication infrastructure between meshes for the overset-HDG method

Out of Scope

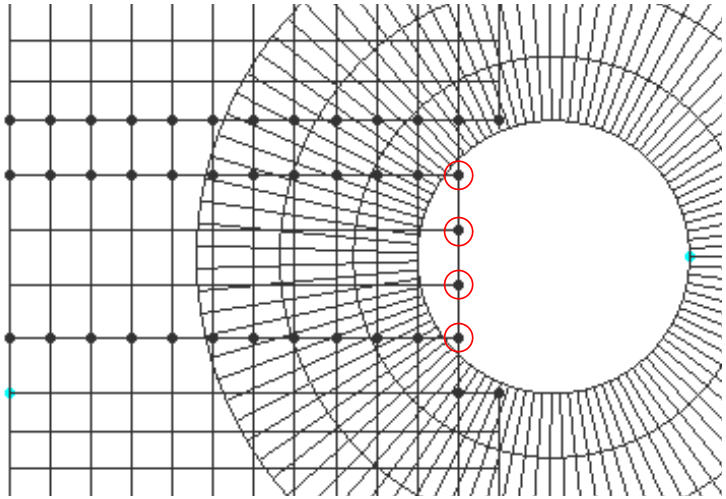
- General overset issues
- Efficiency of HDG

1. Galbraith, Benek, Orkwis, and Turner, *Computer & Fluid*, 2014.
2. Brazell, Sitaraman, and Mavriplis, *Journal of Computational Physics*, 2016.
3. Sheldon, Miller, and Pitt, *Journal of Computational Physics*, 2016.

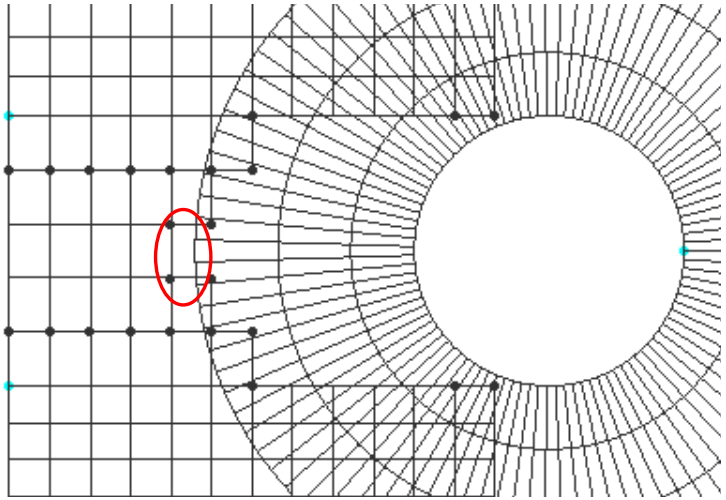
Overset Meshing

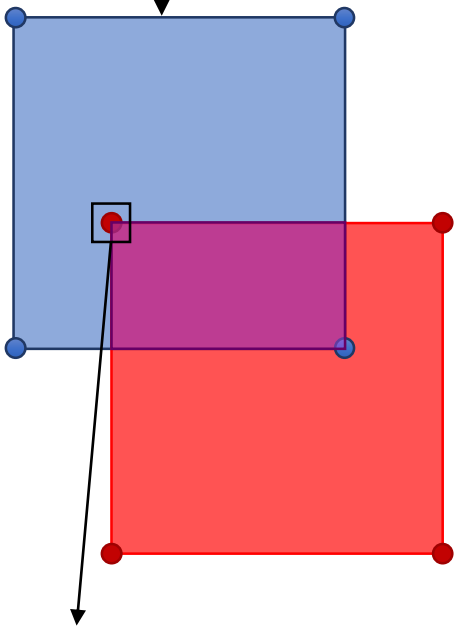
Communication between meshes

Orphan points



Holes





Blue cell donates information to Red node – How?

This is the focus of the coupled overset HDG algorithm!

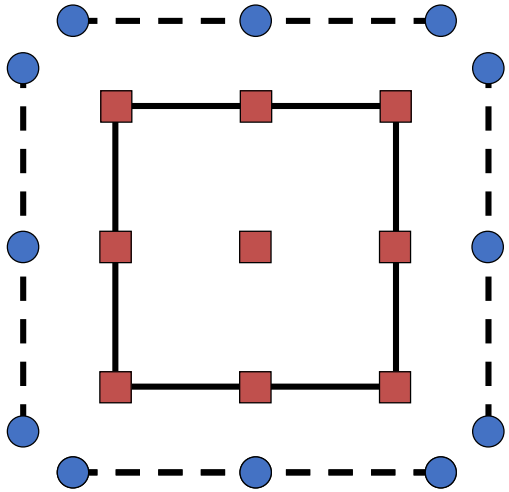
Why HDG?

Continuous Galerkin (CG)	Discontinuous Galerkin (DG)	Hybridizable Discontinuous Galerkin (HDG)
•Structural mechanics •Solution optimization (static condensation) •Well established and robust •Local conservation is not ensured in an Eulerian reference frame[4]	•Robustness of CG and locally conservative like the finite volume method •Efficient scalability of high-order approximations •Large linear system	•Reduces the global linear system through hybridization •Maintains robustness, local conservation, and efficient scalability of high-order approximations •Lack of commercial software •Implementation

4. Hughes and Wells, *Computer Methods in Applied Mechanics and Engineering*, 2005.

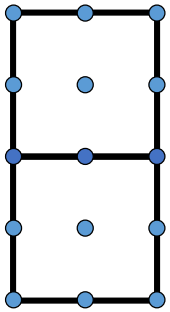
Hybridization

- Independent approximations exist on the border of the elements as well as in the interior of elements for at least one field
- The solution is decomposed into two parts
  - Global solution trace on the cell boundaries
  - Local solution on the interior of each cell

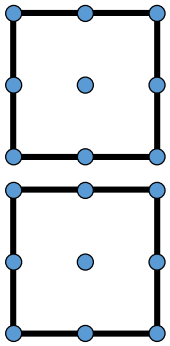


Global DoF Comparison

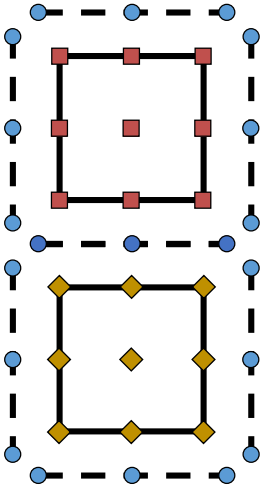
CG



DG



HDG



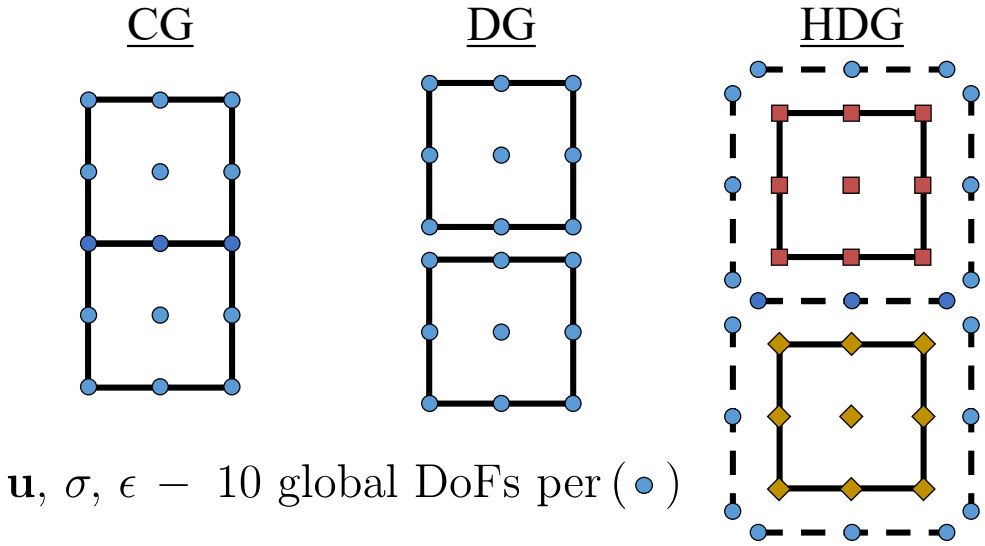
$\mathbf{u}$, σ, ϵ – 10 global DoFs per (●)

$\widehat{\mathbf{u}}$ – 2 global DoFs per (●)

$\mathbf{u}$, σ, ϵ – 10 local DoFs per (■ , ◆)

Unknown	Symbol	CG DoFs per **cell**	DG DoFs per **cell**	HDG DoFs per **face**
Disp.	$\mathbf{u}$, $\widehat{\mathbf{u}}$	$d\,(k+1)^d - s_F[d\,(k+1)]$	$d\,(k+1)^d$	$d\,(k+1)^{d-1}$
Stress	σ	$d^2\,(k+1)^d - s_F[d^2\,(k+1)]$	$d^2\,(k+1)^d$	-
Strain	ϵ	$d^2\,(k+1)^d - s_F[d^2\,(k+1)]$	$d^2\,(k+1)^d$	-

Global DoF Comparison



		CG			DG			HDG		
k		2	3	4	2	3	4	2	3	4
Disp.	$\mathbf{u}$, $\widehat{\mathbf{u}}$	30	56	90	36	64	100	42	58	70
Stress	σ	60	112	180	72	128	200	-	-	-
Strain	ϵ	60	112	180	72	128	200	-	-	-
Total		≤ 150	≤ 280	≤ 450	≤ 180	≤ 320	≤ 500	42	58	70
% change to CG global DoFs					+20%	+14.3%	+11.1%	-72%	-79.3%	-84.4%

There is an 84.4% reduction from CG to HDG

HDG Derivation

Strong Form of Three-Field Elasticity:

$$-\frac{1}{2}\mathbb{C}\left\{\mathrm{Div}\left[\mathrm{Grad}\mathbf{u}+(\mathrm{Grad}\mathbf{u})^{\intercal}\right]\right\}=\mathbf{b}, \quad \forall \mathbf{X}\in\Omega,$$
$$\mathbf{u}=\mathbf{g}_D, \quad \forall \mathbf{X}\in\partial\Omega_D,$$
$$\frac{1}{2}\mathbb{C}\left[\mathrm{Grad}\mathbf{u}+(\mathrm{Grad}\mathbf{u})^{\intercal}\right]\mathbf{n}=\mathbf{t}, \quad \forall \mathbf{X}\in\partial\Omega_N.$$

1st order system →

$$-\mathrm{Div}\sigma-\mathbf{b}=\mathbf{0}, \quad \mathbf{X}\in\Omega,$$
$$\epsilon-\frac{1}{2}\left(\mathrm{Grad}\mathbf{u}+(\mathrm{Grad}\mathbf{u})^{\intercal}\right)=\mathbf{0}, \quad \mathbf{X}\in\Omega,$$
$$\sigma-\mathbb{C}\epsilon=\mathbf{0} \quad \mathbf{X}\in\Omega,$$
$$\mathbf{u}=\overline{\mathbf{u}}, \quad \mathbf{X}\in\partial\Omega_D,$$
$$\sigma\mathbf{n}=\mathbf{t} \quad \mathbf{X}\in\partial\Omega_N$$

Multiply by test functions $\{\mathbf{w},\ \mathbf{C},\ \mathbf{E},\ \omega\}$, integrate over cell K, then integrate by parts

Weak Form Over a Cell:

$$\left(\mathrm{Grad}\mathbf{w},\sigma^{\mathrm{h}}\right)_K-\left\langle\mathbf{w},\widehat{\sigma}^{\mathrm{h}}\mathbf{n}\right\rangle_{\partial K}=\left(\mathbf{w},\mathbf{b}\right)_K,$$
$$\left(\mathbf{C},\epsilon^{\mathrm{h}}\right)_K-\left\langle\frac{1}{2}(\mathbf{C}+\mathbf{C}^{\intercal})\,\mathbf{n},\left(\mathbf{u}^{\mathrm{h}}-\widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial K}$$
$$-\left(\mathbf{C},\frac{1}{2}\left[\mathrm{Grad}\mathbf{u}^{\mathrm{h}}+\left(\mathrm{Grad}\mathbf{u}^{\mathrm{h}}\right)^{\intercal}\right]\right)_K=0,$$
$$\left(\mathbf{E},\sigma^{\mathrm{h}}\right)_K-\left(\mathbf{E},\mathbb{C}\epsilon^{\mathrm{h}}\right)_K=0,$$
$$\left\langle\omega,\widehat{\sigma}^{\mathrm{h}}\mathbf{n}\right\rangle_{\partial K\setminus\partial\Omega_D}=\left\langle\omega,\mathbf{t}\right\rangle_{\partial\Omega_N},$$
$$\left\langle\omega,\widehat{\mathbf{u}}^{\mathrm{h}}\right\rangle_{\partial\Omega_D}=\left\langle\omega,\overline{\mathbf{u}}\right\rangle_{\partial\Omega_D}.$$

What are $\widehat{\mathbf{u}}^{\mathrm{h}}$ and $\widehat{\sigma}^{\mathrm{h}}$?

HDG Derivation

$\widehat{\mathbf{u}}^{\mathrm{h}}$ - Global solution, trace of the hybrid solution

$$\widehat{\sigma}^{\mathrm{h}} := \sigma^{\mathrm{h}} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right) \otimes \mathbf{n}$$

Weak Form of Three-Field Elasticity:

Three-Field Linear Elastostatics

Find $\left(\mathbf{u}^{\mathrm{h}}, \sigma^{\mathrm{h}}, \epsilon^{\mathrm{h}}, \widehat{\mathbf{u}}^{\mathrm{h}}\right) \in \mathscr{U}_{\mathrm{h}} \times \mathscr{C}_{\mathrm{h}} \times \mathscr{E}_{\mathrm{h}} \times \mathscr{M}_{\mathrm{h}}$ *such that*

$$\begin{aligned}
\left(\mathrm{Grad}\mathbf{w}, \sigma^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}} - \left\langle \mathbf{w}, \sigma^{\mathrm{h}}\mathbf{n} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}} &= (\mathbf{w}, \mathbf{b})_{\mathcal{T}_{\mathrm{h}}}\,, \\
\left(\mathbf{C}, \epsilon^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}} - \left\langle \frac{1}{2}\left(\mathbf{C} + \mathbf{C}^{\intercal}\right)\mathbf{n}, \left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}} & \\
- \left(\mathbf{C}, \frac{1}{2}\left[\mathrm{Grad}\mathbf{u}^{\mathrm{h}} + \left(\mathrm{Grad}\mathbf{u}^{\mathrm{h}}\right)^{\intercal}\right]\right)_{\mathcal{T}_{\mathrm{h}}} &= 0, \\
\left(\mathbf{E}, \sigma^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}} - \left(\mathbf{E}, \mathbb{C}\epsilon^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}} &= 0, \\
\left\langle \omega, \sigma^{\mathrm{h}}\mathbf{n} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}\setminus\partial\Omega_D} &= \langle \omega, \mathbf{t}\rangle_{\partial\Omega_N}\,, \\
\left\langle \omega, \widehat{\mathbf{u}}^{\mathrm{h}}\right\rangle_{\partial\Omega_D} &= \langle \omega, \overline{\mathbf{u}}\rangle_{\partial\Omega_D}\,,
\end{aligned}$$

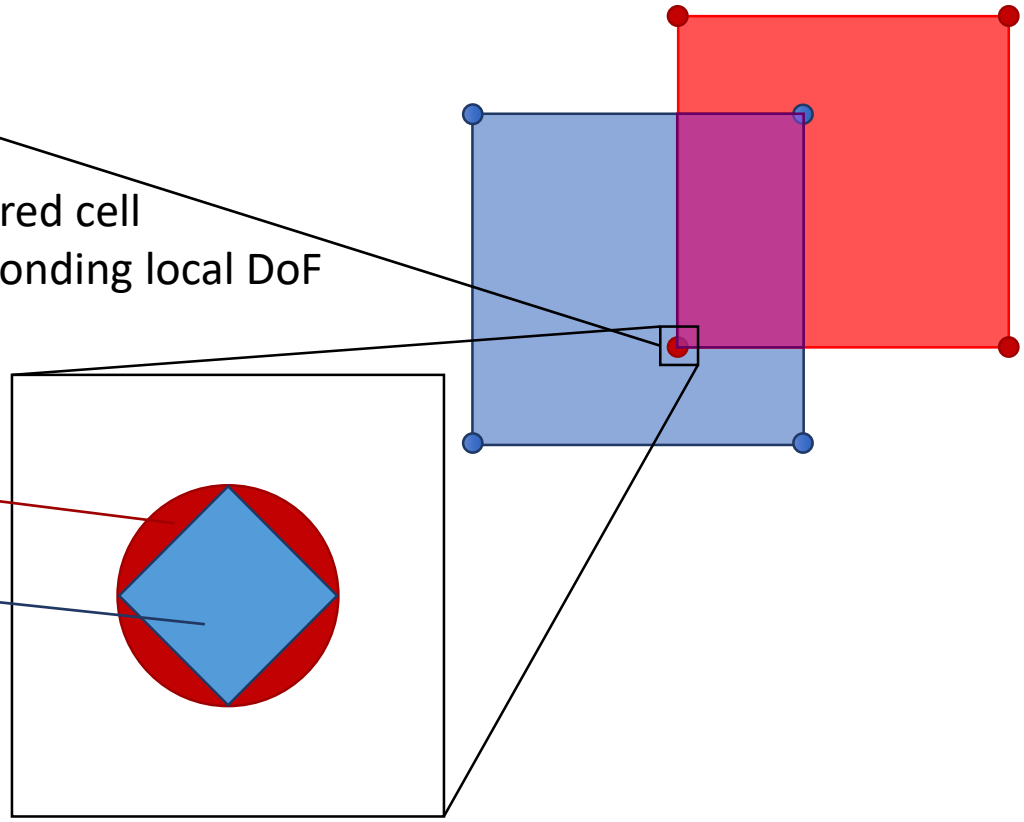
Continuity of numerical trace

$\forall\,(\mathbf{w}, \mathbf{C}, \mathbf{E}, \omega) \in \mathscr{U}_{\mathrm{h}} \times \mathscr{C}_{\mathrm{h}} \times \mathscr{E}_{\mathrm{h}} \times \mathscr{M}_{\mathrm{h}}.$

Overset HDG

● = global DoFs

Communication occurs between the global DoF from the red cell at this location and a new corresponding local DoF from the blue cell at this location.



$\widehat{\mathbf{u}}^{\mathrm{h}}$

$\mathbf{u}^{\mathrm{h}},\ \sigma^{\mathrm{h}},\ \epsilon^{\mathrm{h}}$

If this location was not an overset boundary the resulting term would be included in

$$\left\langle \omega, \sigma^{\mathrm{h}}\mathbf{n} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}\setminus\partial\Omega_D}$$

since there would be a neighboring cell to communicate with.

Instead $\longrightarrow$ $\left\langle \omega, \breve{\sigma}^{\mathrm{h}}\breve{\mathbf{n}} - \tau\left(\breve{\mathbf{u}}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\Omega_O^i}$

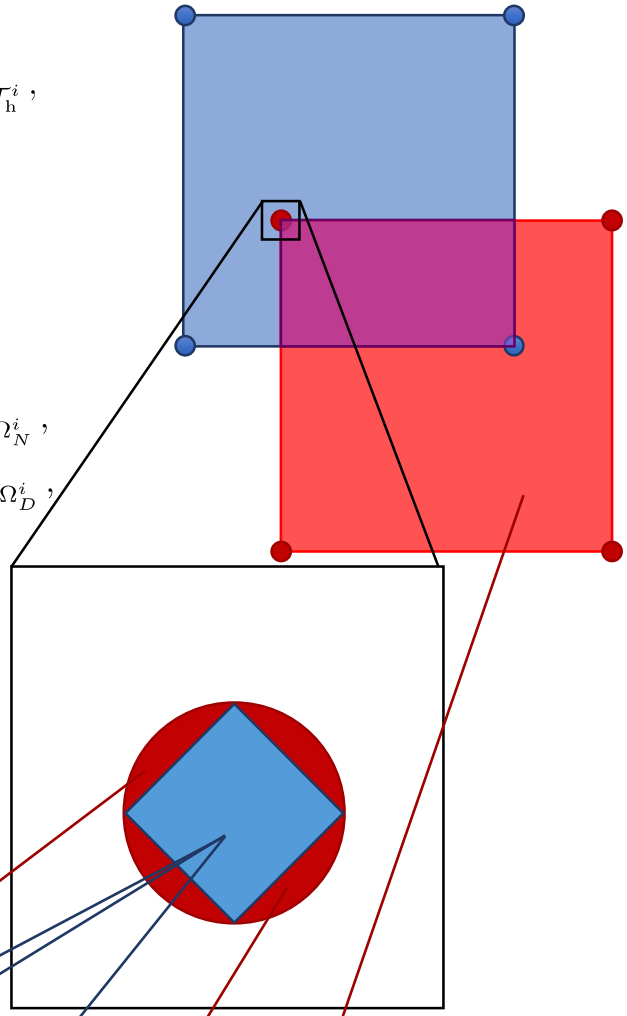
Three-Field Elasticity

Overset Three-Field Elasticity

Find $\left(\mathbf{u}^{\mathrm{h}}, \sigma^{\mathrm{h}}, \epsilon^{\mathrm{h}}, \widehat{\mathbf{u}}^{\mathrm{h}}\right) \in \mathscr{U}_{\mathrm{h}}^{i} \times \mathscr{C}_{\mathrm{h}}^{i} \times \mathscr{E}_{\mathrm{h}}^{i} \times \mathscr{M}_{\mathrm{h}}^{i}$ *such that*

$$
\begin{aligned}
\left(\mathrm{Grad}\mathbf{w}, \sigma^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left\langle \mathbf{w}, \sigma^{\mathrm{h}}\mathbf{n} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} &= \left(\mathbf{w}, \mathbf{b}\right)_{\mathcal{T}_{\mathrm{h}}^{i}}, \\
\left(\mathbf{C}, \epsilon^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left\langle \frac{1}{2}(\mathbf{C} + \mathbf{C}^{\intercal})\,\mathbf{n}, \left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} & \\
- \left(\mathbf{C}, \frac{1}{2}\left[\mathrm{Grad}\mathbf{u}^{\mathrm{h}} + \left(\mathrm{Grad}\mathbf{u}^{\mathrm{h}}\right)^{\intercal}\right]\right)_{\mathcal{T}_{\mathrm{h}}^{i}} &= 0, \\
\left(\mathbf{E}, \sigma^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left(\mathbf{E}, \mathbb{C}\epsilon^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} &= 0, \\
\left\langle \omega, \sigma^{\mathrm{h}}\mathbf{n} - \tau\left(\mathbf{u}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}\setminus\partial\Omega_{D}^{i}} + \boxed{\left\langle \omega, \breve{\sigma}^{\mathrm{h}}\breve{\mathbf{n}} - \tau\left(\breve{\mathbf{u}}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\Omega_{O}^{i}}} &= \left\langle \omega, \mathbf{t}\right\rangle_{\partial\Omega_{N}^{i}}, \\
\left\langle \omega, \widehat{\mathbf{u}}^{\mathrm{h}}\right\rangle_{\partial\Omega_{D}^{i}} &= \left\langle \omega, \overline{\mathbf{u}}\right\rangle_{\partial\Omega_{D}^{i}},
\end{aligned}
$$

$\forall\,(\mathbf{w}, \mathbf{C}, \mathbf{E}, \omega) \in \mathscr{U}_{\mathrm{h}}^{i} \times \mathscr{C}_{\mathrm{h}}^{i} \times \mathscr{E}_{\mathrm{h}}^{i} \times \mathscr{M}_{\mathrm{h}}^{i}.$



$$\left\langle \omega, \breve{\sigma}^{\mathrm{h}}\breve{\mathbf{n}} - \tau\left(\breve{\mathbf{u}}^{\mathrm{h}} - \widehat{\mathbf{u}}^{\mathrm{h}}\right)\right\rangle_{\partial\Omega_{O}^{i}}$$

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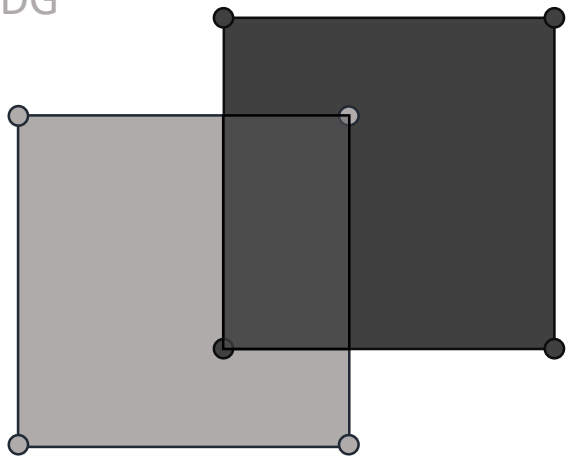

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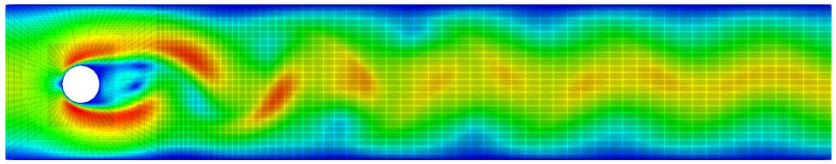

  - Arbitrary Lagrangian-Eulerian (ALE) Navier-Stokes

- Future Work
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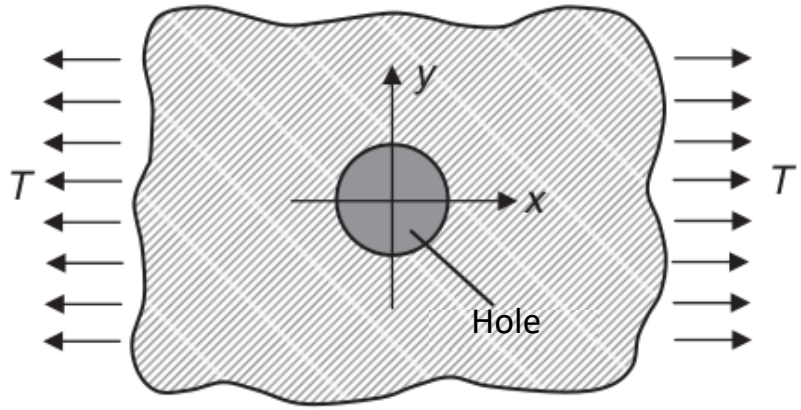
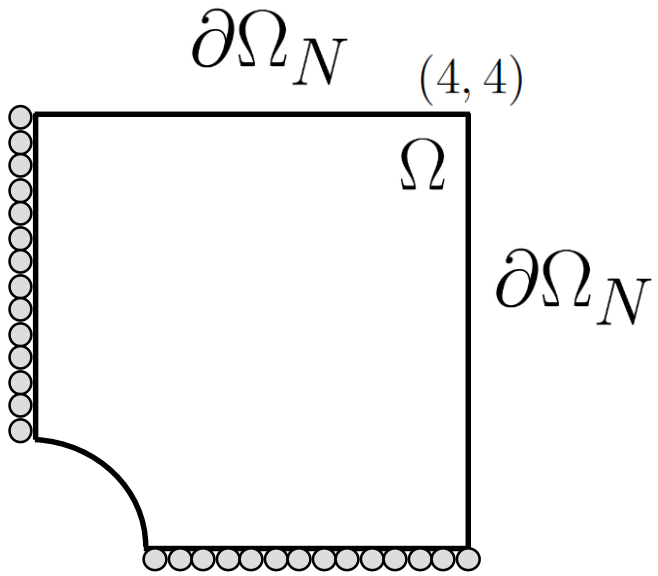
Three-Field Application

Compared to an analytical solution



$T = 1\,N$



$$r := \sqrt{x^2 + y^2} \qquad \theta = \arctan\frac{y}{x} \qquad a = 1\,m$$

Apply traction boundary conditions at $x = 4$ and $y = 4$ since the exact stress field is

$$\sigma^e_{xx} = 1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3}{2}\frac{a^4}{r^4}\cos 4\theta,$$

$$\sigma^e_{yy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3}{2}\frac{a^4}{r^4}\cos 4\theta,$$

$$\sigma^e_{xy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) + \frac{3}{2}\frac{a^4}{r^4}\sin 4\theta,$$

$E = 10^3\,Pa \qquad \nu = 0.3$

The Lamé constants are

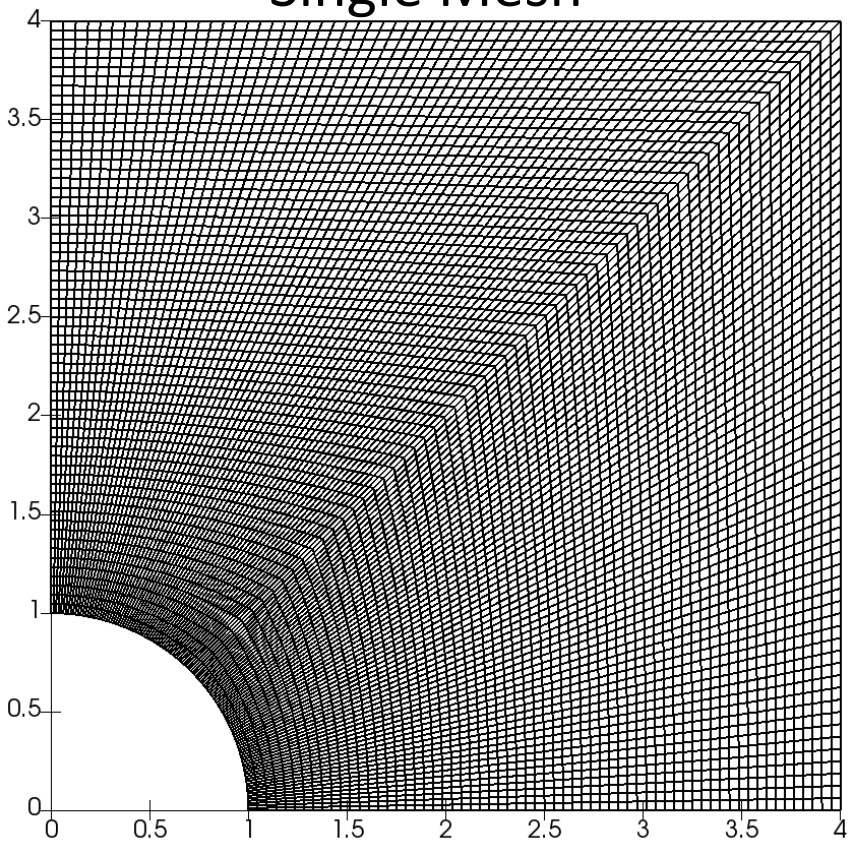
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(2+\nu)}$$

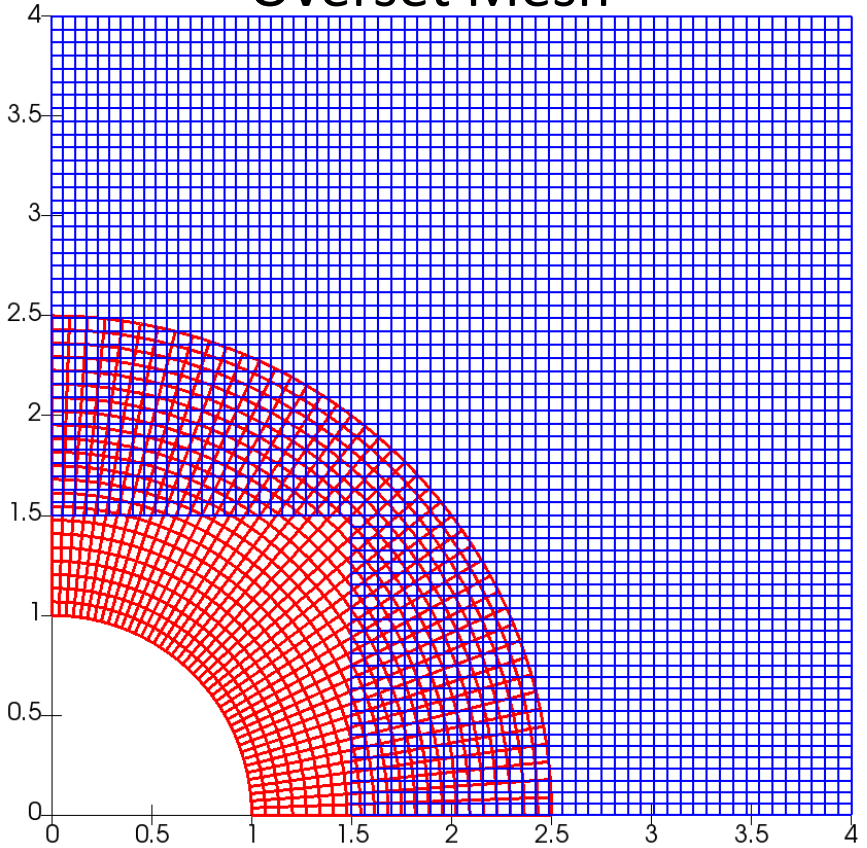
$\tau := \mu\mathbf{I}$

Meshes

Single Mesh



Overset Mesh



8,192 Cells

198,912 global DoFs

4,388 total Cells

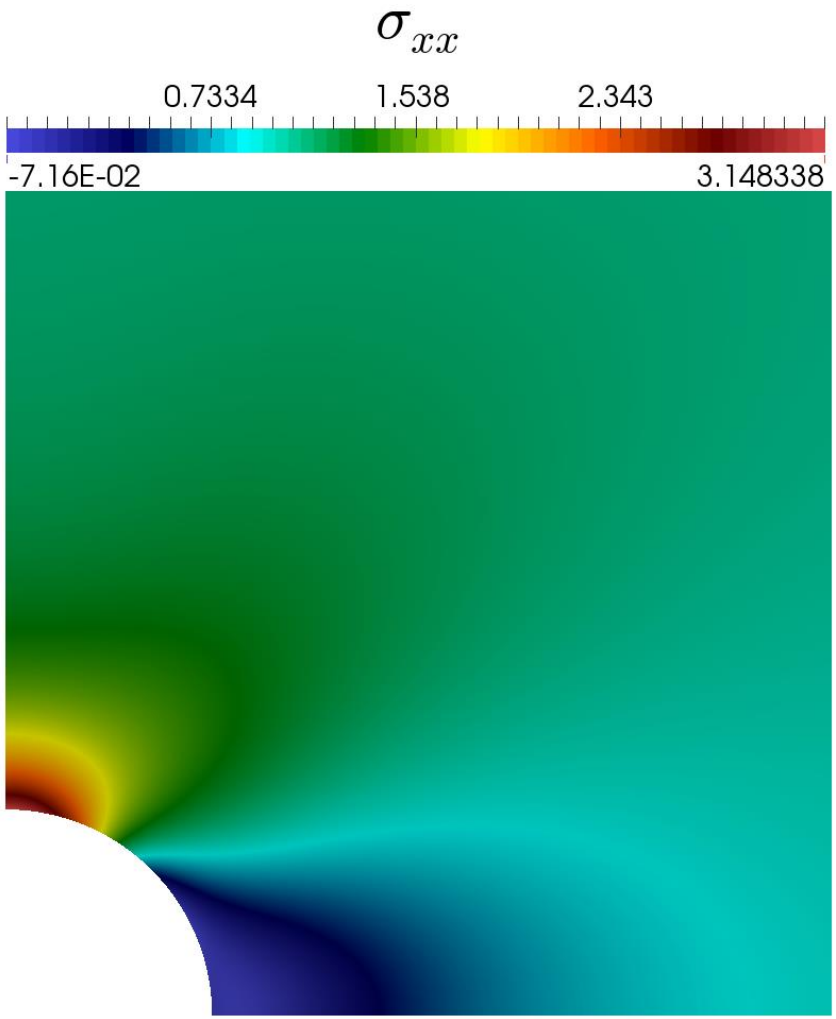
107,640 total global DoFs

46.44% Cell reduction

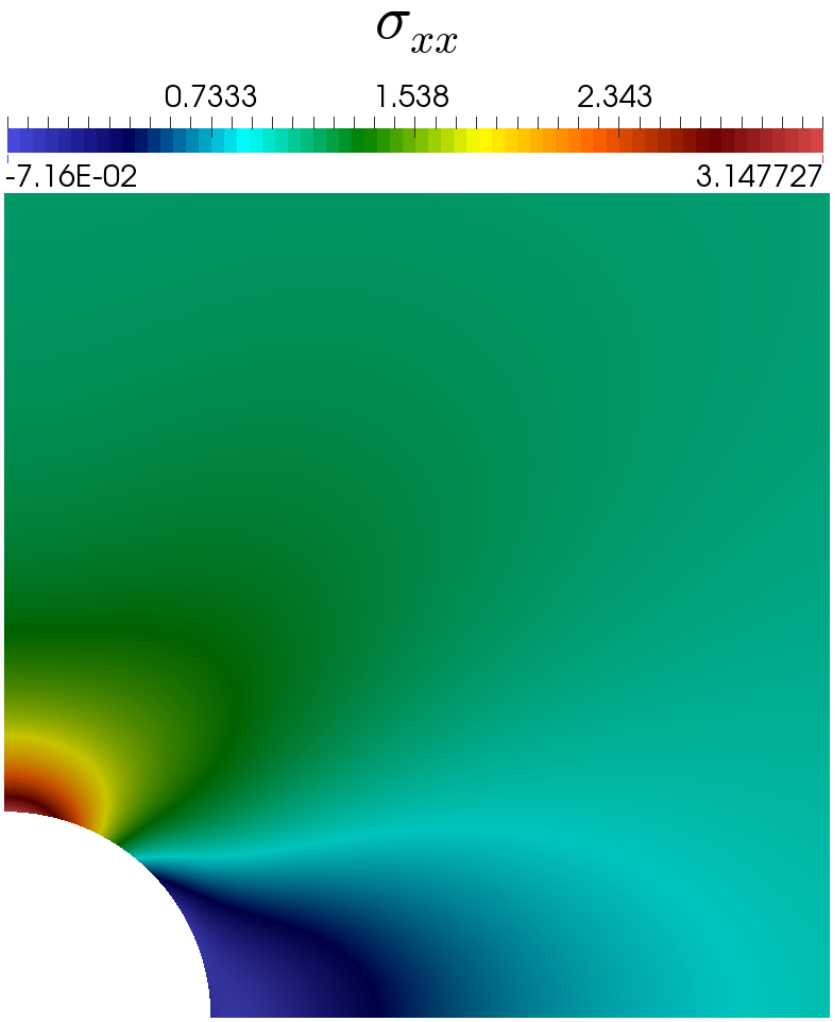
45.89% global DoFs reduction

Q5 elements

Results

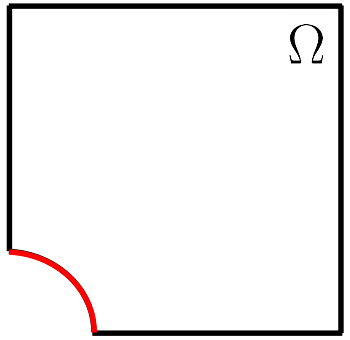


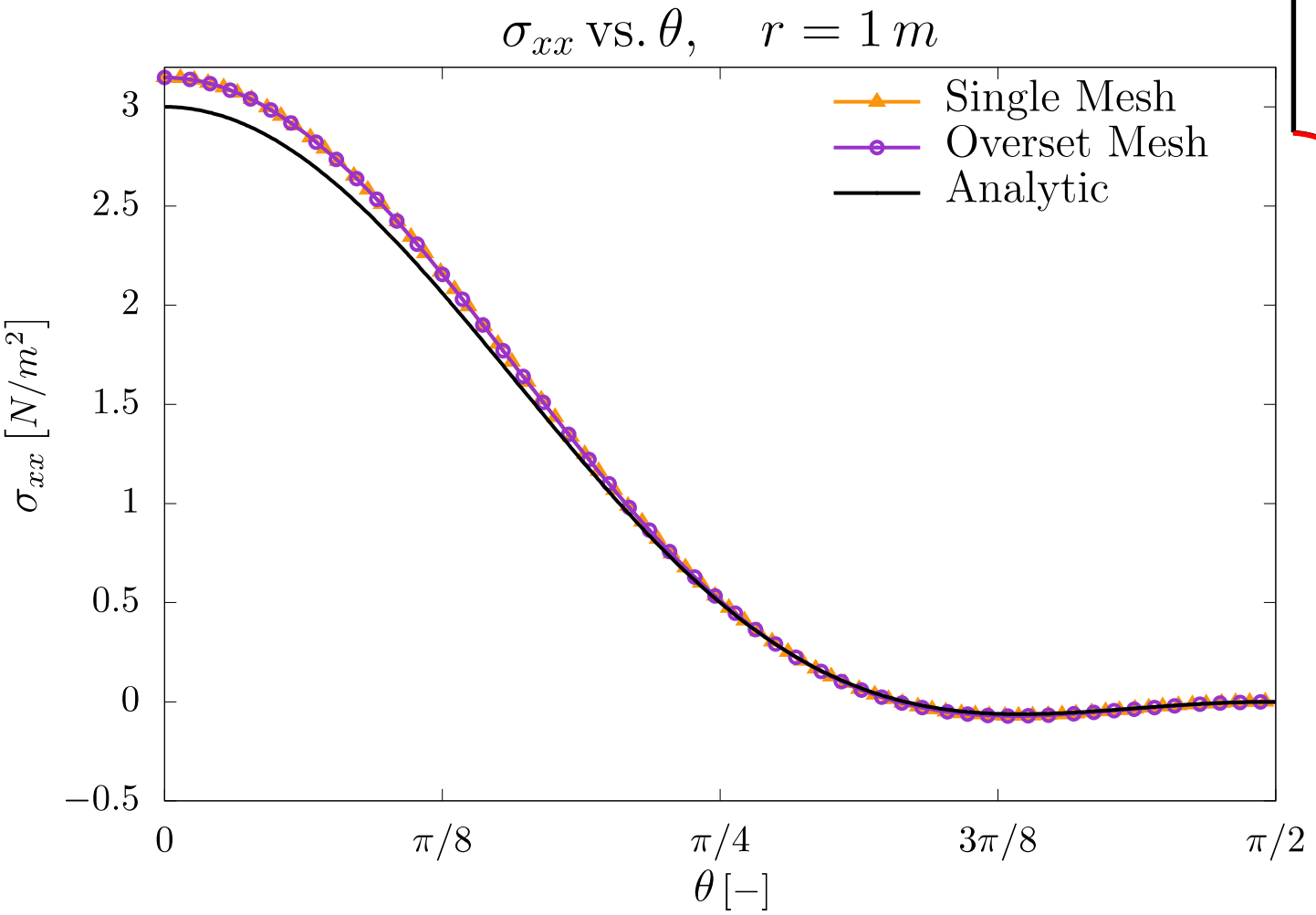
Single Mesh



Overset Mesh

Comparison Plots





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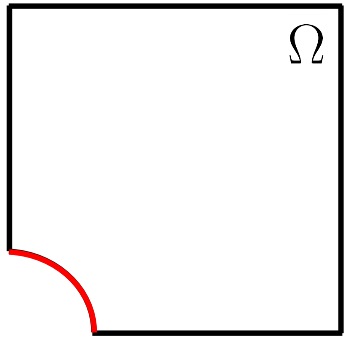
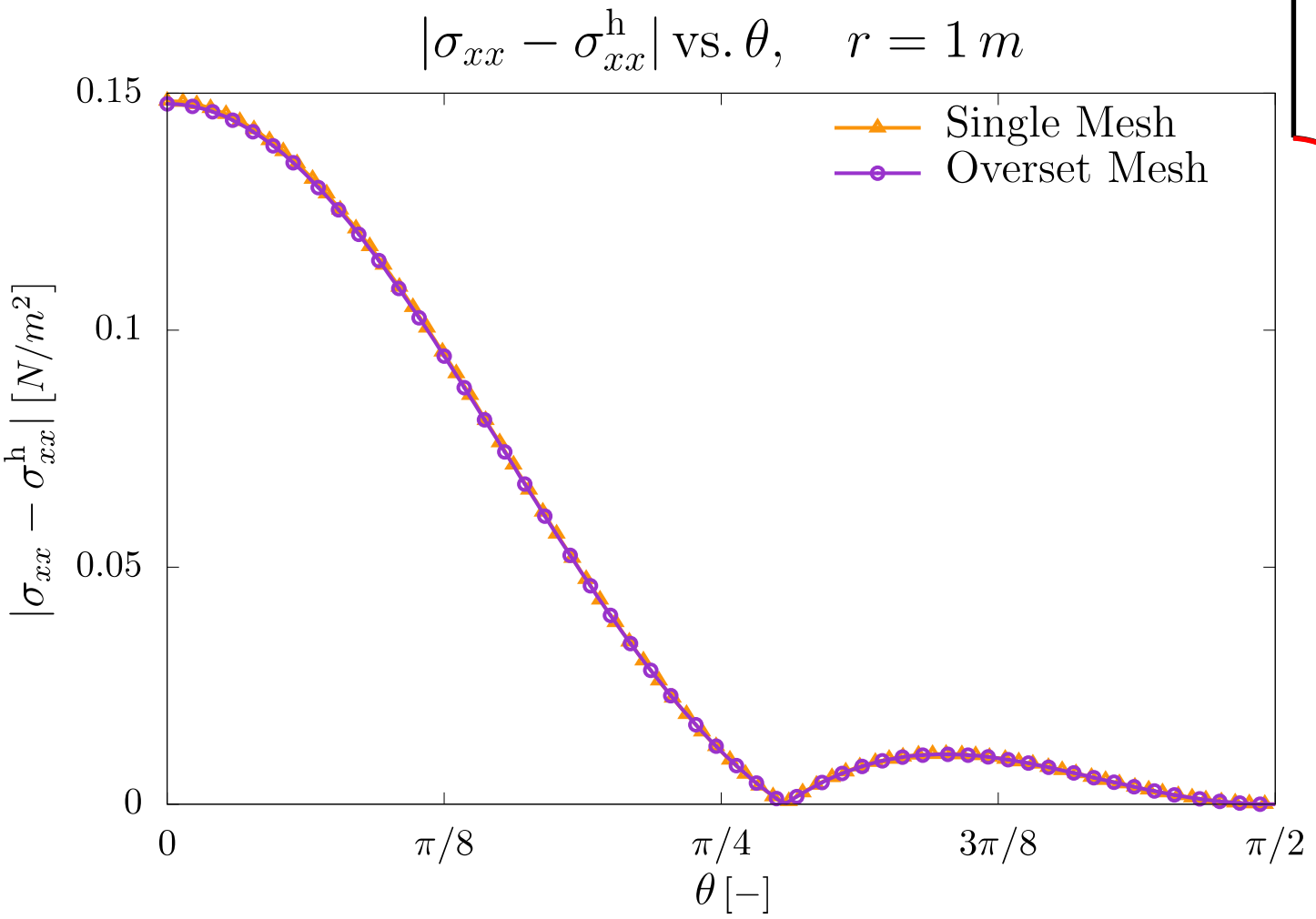
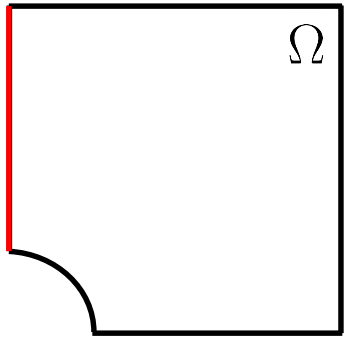
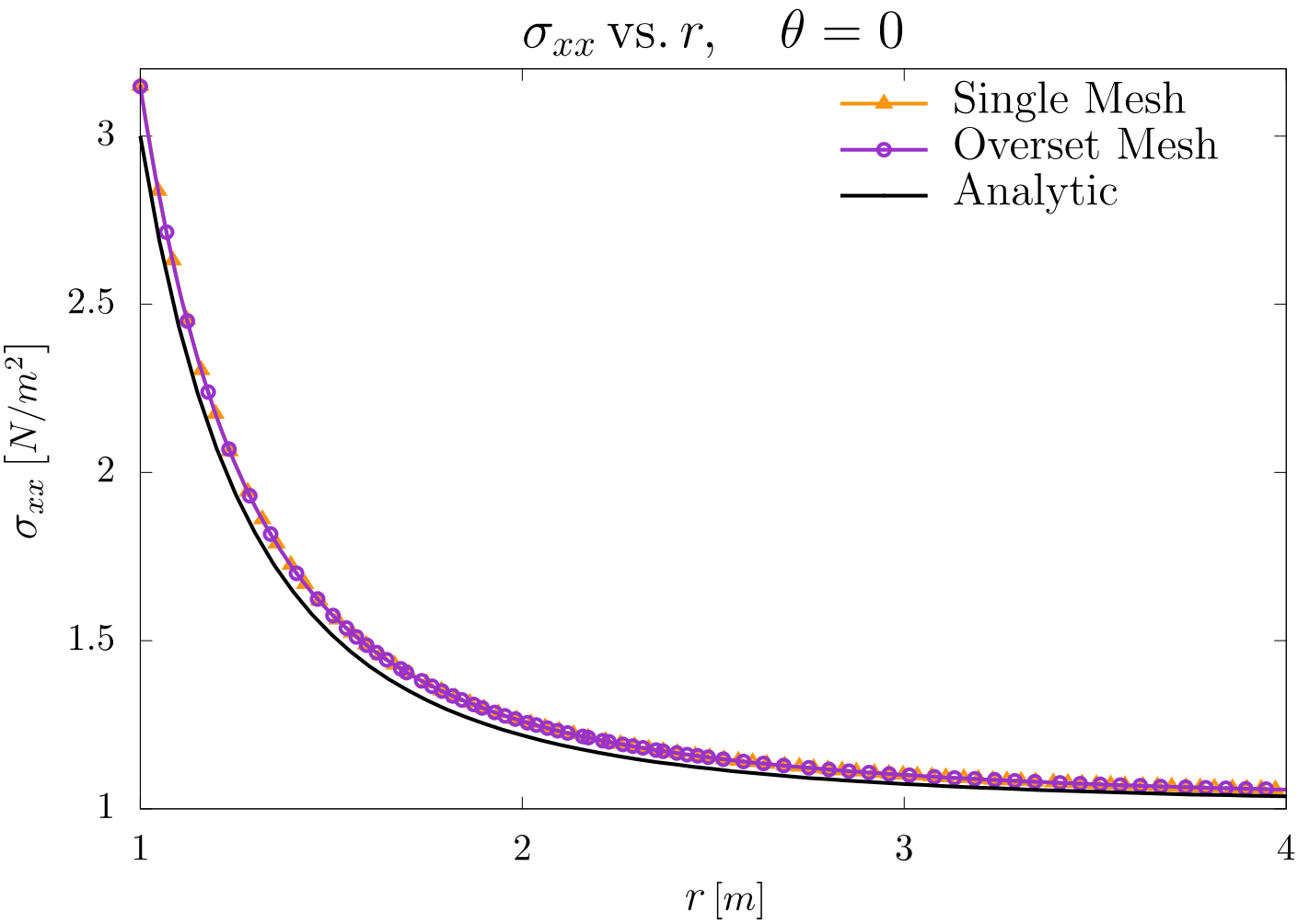


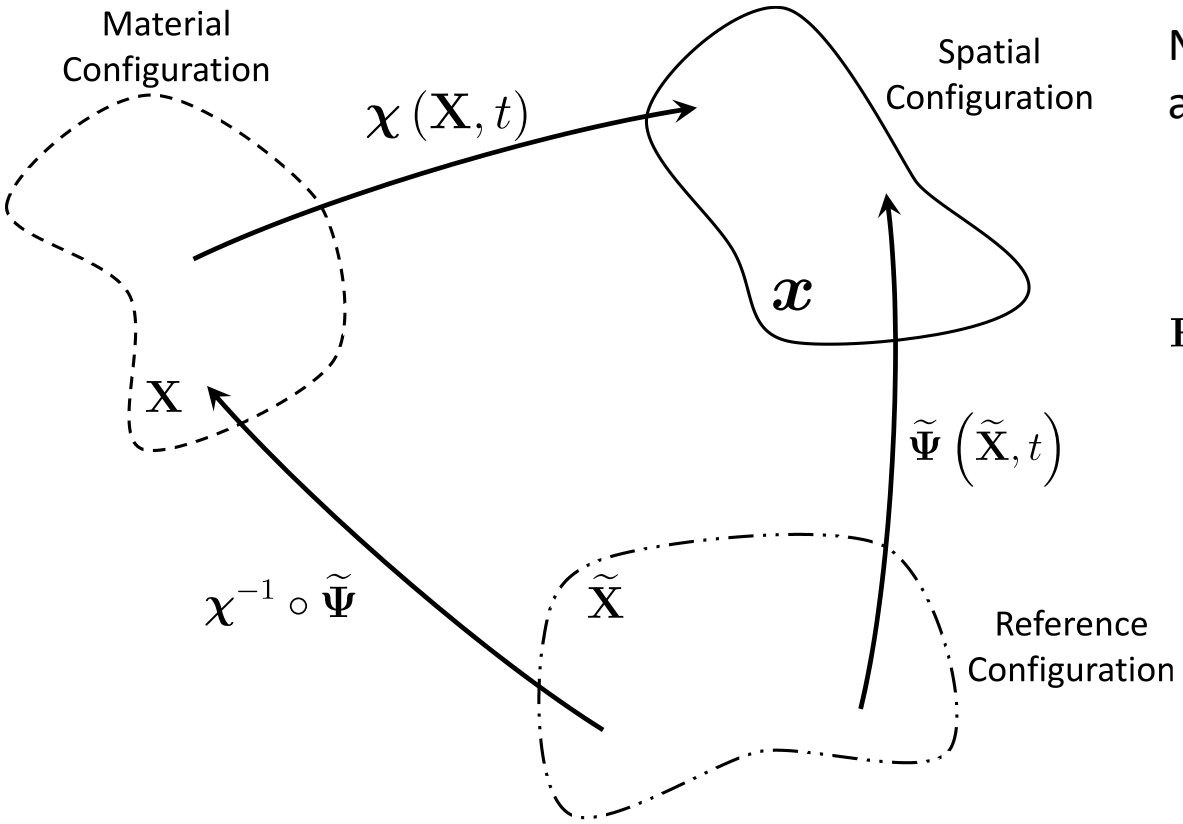


Comparison Plots





ALE Formulation



Necessary transformations and definitions

$$\mathbf{F}_{\mathrm{m}} := \frac{\partial \widetilde{\boldsymbol{\Psi}}\left(\widetilde{\mathbf{X}}, t\right)}{\partial \widetilde{\mathbf{X}}} := \mathbf{I} + \mathrm{Grad}\,\mathbf{u}_{\mathrm{m}}$$

$$\begin{aligned}
\mathrm{grad}\,p &= \mathbf{F}_{\mathrm{m}}^{-\mathsf{T}} \mathrm{Grad}\,p_{\mathrm{f}}, \\
\mathrm{grad}\,\mathbf{v} &= \left(\mathrm{Grad}\,\mathbf{v}_{\mathrm{f}}\right) \mathbf{F}_{\mathrm{m}}^{-1}, \\
\mathrm{div}\,\mathbf{L} &= \mathrm{Grad}\,\mathbf{L}_{\mathrm{f}} : \mathbf{F}_{\mathrm{m}}^{-\mathsf{T}}, \\
\mathrm{div}\,\mathbf{v} &= \mathrm{Grad}\,\mathbf{v}_{\mathrm{f}} : \mathbf{F}_{\mathrm{m}}^{-\mathsf{T}}.
\end{aligned}$$

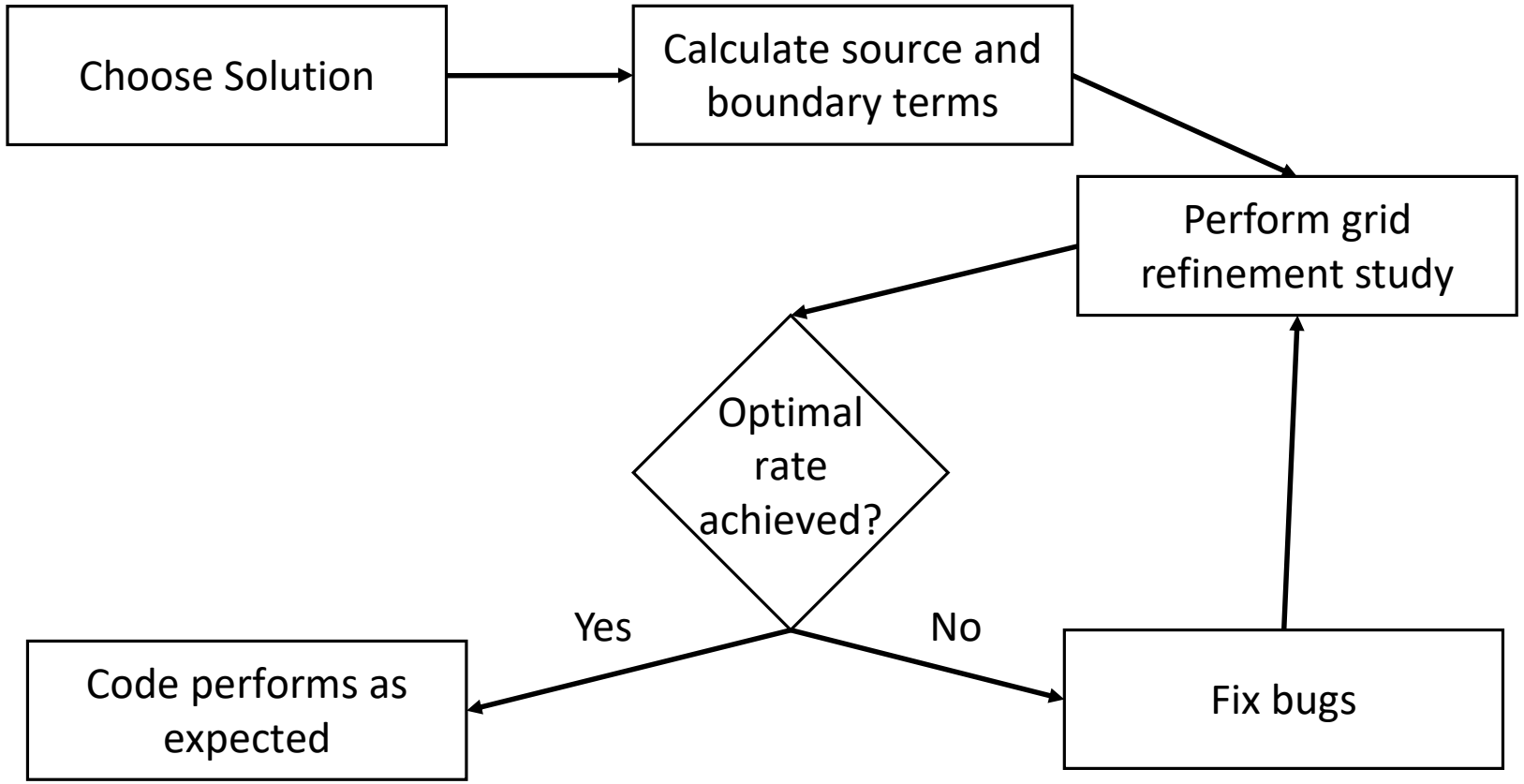
$$dv = J_{\mathrm{m}}\, dv_{\mathrm{m}}, \quad da = J_{\mathrm{m}}\, da_{\mathrm{m}},$$
$$\mathbf{n}\, da = \mathbf{F}_{\mathrm{m}}^{C} \mathbf{n}_{\mathrm{f}}\, da_{\mathrm{m}} = J_{\mathrm{m}} \mathbf{F}_{\mathrm{m}}^{-\mathsf{T}} \mathbf{n}_{\mathrm{f}}\, da_{\mathrm{m}}.$$

$$\phi' = \dot{\phi} - \mathbf{F}_{\mathrm{m}}^{-\mathsf{T}} \mathrm{Grad}\,\phi \cdot \frac{\partial \widetilde{\mathbf{X}}}{\partial t},$$

$$\mathbf{a}' = \dot{\mathbf{a}} - \left[\left(\mathrm{Grad}\,\mathbf{a}\right) \mathbf{F}_{\mathrm{m}}^{-1}\right] \frac{\partial \widetilde{\mathbf{X}}}{\partial t}.$$

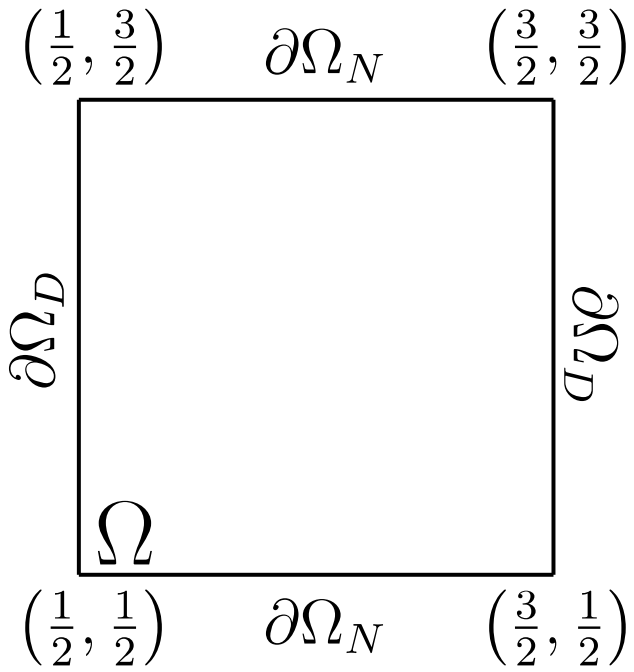
MMS

The method of manufactured solutions (MMS) provides code verification for each problem being solve.



Elastostatics Verification

$$
\begin{aligned}
\mathbf{F} - \mathbf{I} - \mathrm{Grad}\,\mathbf{u} &= \mathbf{0}, && \text{in } \Omega, \\
-\frac{1}{2}\mathbb{C}\left[\mathrm{Div}\left\{[\mathbf{F}-\mathbf{I}] + [\mathbf{F}-\mathbf{I}]^{\intercal}\right\}\right] &= \mathbf{b}, && \text{in } \Omega, \\
\mathbf{u} &= \mathbf{g}_D, && \text{on } \partial\Omega_D, \\
\frac{1}{2}\mathbb{C}\left\{[\mathbf{F}-\mathbf{I}] + [\mathbf{F}-\mathbf{I}]^{\intercal}\right\}\mathbf{n} &= \mathbf{t}, && \text{on } \partial\Omega_N.
\end{aligned}
$$



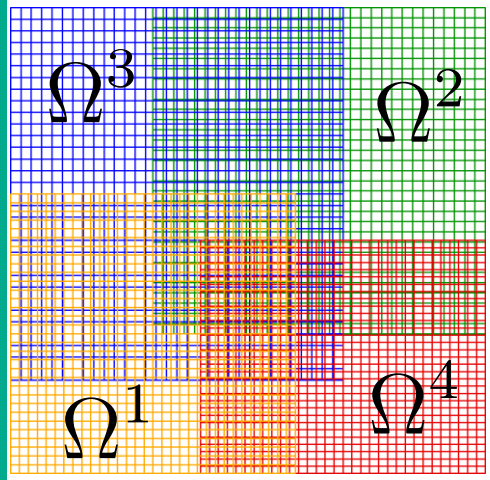
$$
\mathbf{u}_{exact} = \begin{pmatrix} -x - \sin\left(\frac{\pi}{4}\left[x-y\right]\right)\cos\left(\frac{\pi}{4}t\right) \\ -y + \cos\left(\frac{\pi}{4}\left[x+y\right]\right)\cos\left(\frac{\pi}{4}t\right) \end{pmatrix}
$$

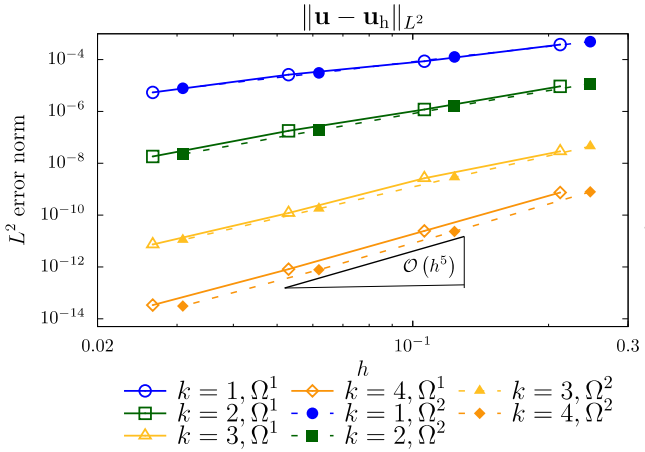
$$
\mu = 0.5\ Pa \qquad \lambda = 1\ Pa \qquad \Delta t = 10^{-8}\,s
$$

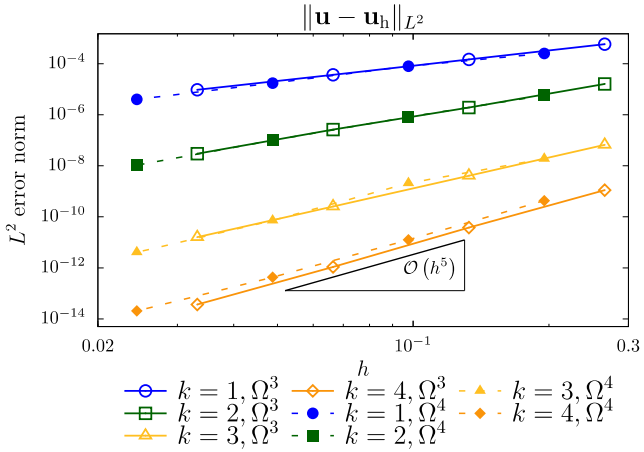
$$
\mathbf{b} = \begin{pmatrix} \frac{\pi^2}{16}\cos\left(\frac{\pi}{4}t\right)\left\{(\lambda+\mu)\cos\left(\frac{\pi}{4}\left[x+y\right]\right) - (\lambda+3\mu)\sin\left(\frac{\pi}{4}\left[x-y\right]\right)\right\} \\ \frac{\pi^2}{16}\cos\left(\frac{\pi}{4}t\right)\left\{(\lambda+\mu)\sin\left(\frac{\pi}{4}\left[x-y\right]\right) + (\lambda+3\mu)\cos\left(\frac{\pi}{4}\left[x+y\right]\right)\right\} \end{pmatrix}
$$

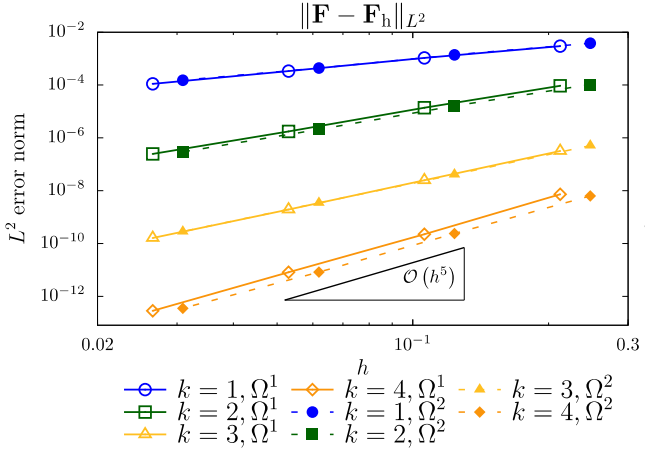
$$
\mathbf{S} = 10.5\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}
$$

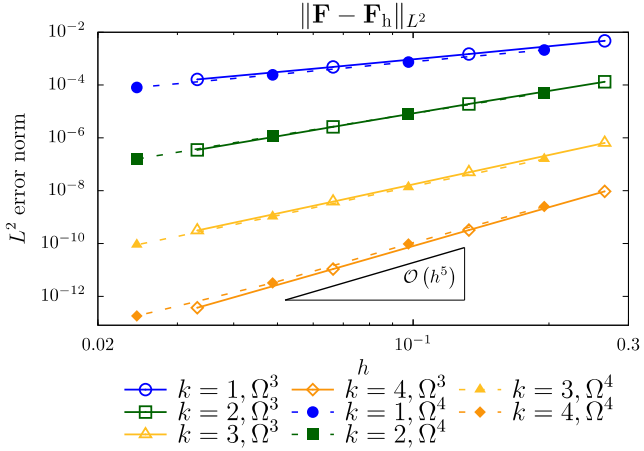
Verification Results











Navier-Stokes

Strong Form of Navier-Stokes:

$$\mathbf{L} - \operatorname{grad}\mathbf{v} = \mathbf{0}, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}\left[-\mu\mathbf{L} + p\mathbf{I} + \rho\left(\mathbf{v}\otimes\mathbf{v}\right)\right] = \mathbf{f}, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\frac{\partial \rho}{\partial t} + \left(\operatorname{grad}\rho\right)\mathbf{v} + \rho\operatorname{div}\mathbf{v} = 0, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\mathbf{v} = \mathbf{v}_D, \qquad \forall \boldsymbol{x} \in \partial\Omega_D,$$

$$\left(-\mu\mathbf{L} + p\mathbf{I}\right)\mathbf{n} = \mathbf{g}_N, \qquad \forall \boldsymbol{x} \in \partial\Omega_N.$$

Assume density is constant spatially

$$\operatorname{div}\left[\rho\left(\mathbf{v}\otimes\mathbf{v}\right)\right] = \rho\operatorname{div}\left(\mathbf{v}\right)\mathbf{v} + \rho\operatorname{grad}\left(\mathbf{v}\right)\mathbf{v} + \left[\mathbf{v}\otimes\operatorname{grad}\rho\right]\mathbf{v}$$

$\operatorname{grad}\rho \to 0$, $\operatorname{div}\mathbf{v} \to 0$, $\operatorname{grad}\left(\mathbf{v}\right)\mathbf{v} \to 0$

Assume entropy is constant[5]

$$\frac{\partial \rho}{\partial t} = \beta\frac{\partial p}{\partial t} \qquad \beta := 1/c^2$$

Strong Form of Isentropic Navier-Stokes:

$$\mathbf{L} - \operatorname{grad}\mathbf{v} = \mathbf{0}, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}\left(-\mu\mathbf{L} + p\mathbf{I}\right) + \rho\mathbf{L}\mathbf{v} + \rho\left(\operatorname{div}\mathbf{v}\right)\mathbf{v} = \mathbf{f}, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\varepsilon\frac{\partial p}{\partial t} + \operatorname{div}\mathbf{v} = 0, \qquad \forall \boldsymbol{x} \in \Omega,$$

$$\mathbf{v} = \mathbf{g}_D, \qquad \forall \boldsymbol{x} \in \partial\Omega_D,$$

$$\left(-\mu\mathbf{L} + p\mathbf{I}\right)\mathbf{n} = \mathbf{g}_N, \qquad \forall \boldsymbol{x} \in \partial\Omega_N.$$

$$\varepsilon := \beta/\rho$$

Changes from an incompressible formulation

Flows of interest are low Mach number.[5,6]

5. Inagaki, Murata, and Kondoh, *AIAA Journal*, 2002.
6. Klainerman and Majda, *Communications on pure and applied Mathematics*, 1981

Isentropic Navier-Stokes

Weak Form of Isentropic Navier-Stokes:

Overset Isentropic Navier-Stokes

Find $\left(\mathbf{L}^{\mathrm{h}}, \mathbf{v}^{\mathrm{h}}, p^{\mathrm{h}}, \widehat{\mathbf{v}}^{\mathrm{h}}\right) \in \mathscr{G}_{\mathrm{h}}^{i} \times \mathscr{Y}_{\mathrm{h}}^{i} \times \mathscr{P}_{\mathrm{h}}^{i} \times \mathscr{N}_{\mathrm{h}}^{i}$ *such that*

$$
\begin{aligned}
\left(\mathbf{G}, \operatorname{grad}\mathbf{v}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left(\mathbf{G}, \mathbf{L}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left\langle \mathbf{G}, \mathbf{v}^{\mathrm{h}} \otimes \mathbf{n} - \widehat{\mathbf{v}}^{\mathrm{h}} \otimes \mathbf{n} \right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} &= 0, \\
\left(\mathbf{y}, \rho \frac{\mathbf{v}^{\mathrm{h}}}{\partial t}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} + \left(\operatorname{grad}\mathbf{y}, \mu\mathbf{L}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left(\operatorname{div}\mathbf{y}, p^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} & \\
+ \left(\mathbf{y}, \rho\mathbf{L}^{\mathrm{h}}\mathbf{v}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} + \left(\mathbf{y}, \rho\left[\operatorname{div}\mathbf{v}^{\mathrm{h}}\right]\mathbf{v}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} + \left\langle \mathbf{y}, \widehat{\mathbf{T}}^{\mathrm{h}}\mathbf{n} \right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} &= \left(\mathbf{y}, \mathbf{f}\right)_{\mathcal{T}_{\mathrm{h}}^{i}}, \\
\left(q, \varepsilon \frac{\partial p^{\mathrm{h}}}{\partial t}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} - \left(\operatorname{grad}q, \mathbf{v}^{\mathrm{h}}\right)_{\mathcal{T}_{\mathrm{h}}^{i}} + \left\langle q, \widehat{\mathbf{v}}^{\mathrm{h}} \cdot \mathbf{n} \right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} &= \left(q, \gamma\right)_{\mathcal{T}_{\mathrm{h}}^{i}}, \\
\left\langle \boldsymbol{\eta}, \widehat{\mathbf{v}}^{\mathrm{h}} \right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}} &= \left\langle \boldsymbol{\eta}, \mathbf{g}_{D} \right\rangle_{\partial\Omega_{D}^{i}}, \\
\left\langle \boldsymbol{\eta}, \widehat{\mathbf{T}}^{\mathrm{h}}\mathbf{n} \right\rangle_{\partial\mathcal{T}_{\mathrm{h}}^{i}\setminus\partial\Omega_{D}^{i}} + \boxed{\left\langle \boldsymbol{\eta}, \breve{\mathbf{T}}^{\mathrm{h}}\breve{\mathbf{n}} \right\rangle_{\partial\Omega_{O}^{i}}} &= \left\langle \boldsymbol{\eta}, \mathbf{g}_{N} \right\rangle_{\partial\Omega_{N}^{i}},
\end{aligned}
$$

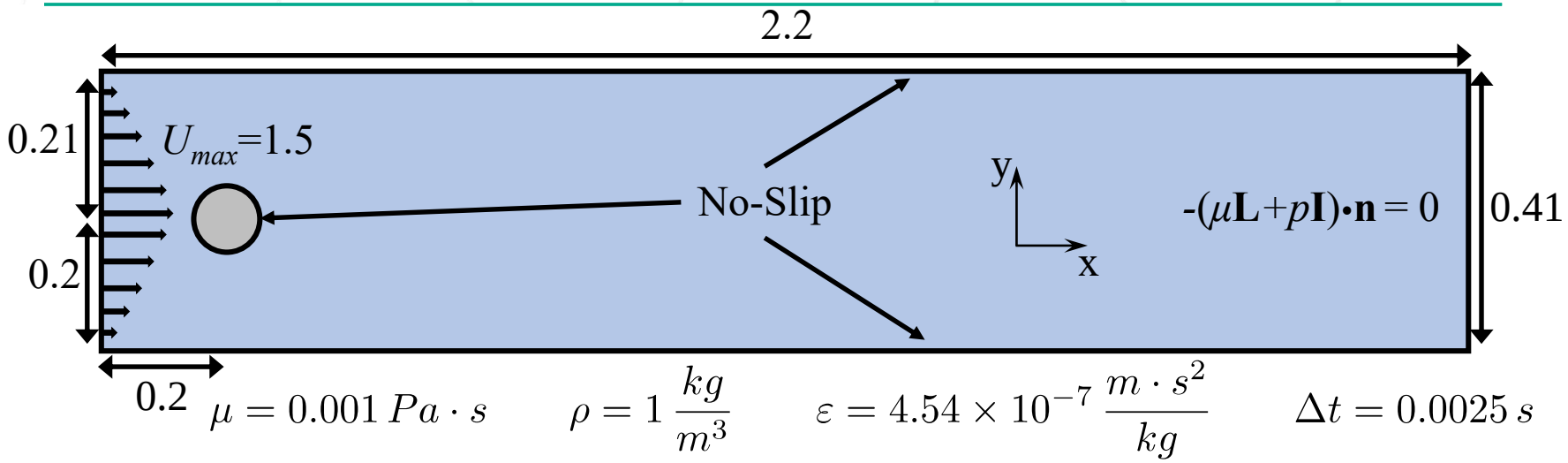
$\forall \left(\mathbf{G}, \mathbf{y}, q, \boldsymbol{\eta}\right) \in \mathscr{G}_{\mathrm{h}}^{i} \times \mathscr{Y}_{\mathrm{h}}^{i} \times \mathscr{P}_{\mathrm{h}}^{i} \times \mathscr{N}_{\mathrm{h}}^{i}$ *where*

$$
\begin{aligned}
\widehat{\mathbf{T}}^{\mathrm{h}}\mathbf{n} &= -\mu\mathbf{L}^{\mathrm{h}}\mathbf{n} + p^{\mathrm{h}}\mathbf{n} + \mathbf{S}_{\mathrm{f}}\left(\mathbf{v}^{\mathrm{h}} - \widehat{\mathbf{v}}^{\mathrm{h}}\right), \\
\boxed{\breve{\mathbf{T}}^{\mathrm{h}}\breve{\mathbf{n}} &= -\mu\breve{\mathbf{L}}^{\mathrm{h}}\breve{\mathbf{n}} + \breve{p}^{\mathrm{h}}\breve{\mathbf{n}} + \mathbf{S}_{\mathrm{f}}\left(\breve{\mathbf{v}}^{\mathrm{h}} - \widehat{\mathbf{v}}^{\mathrm{h}}\right)}.
\end{aligned}
$$

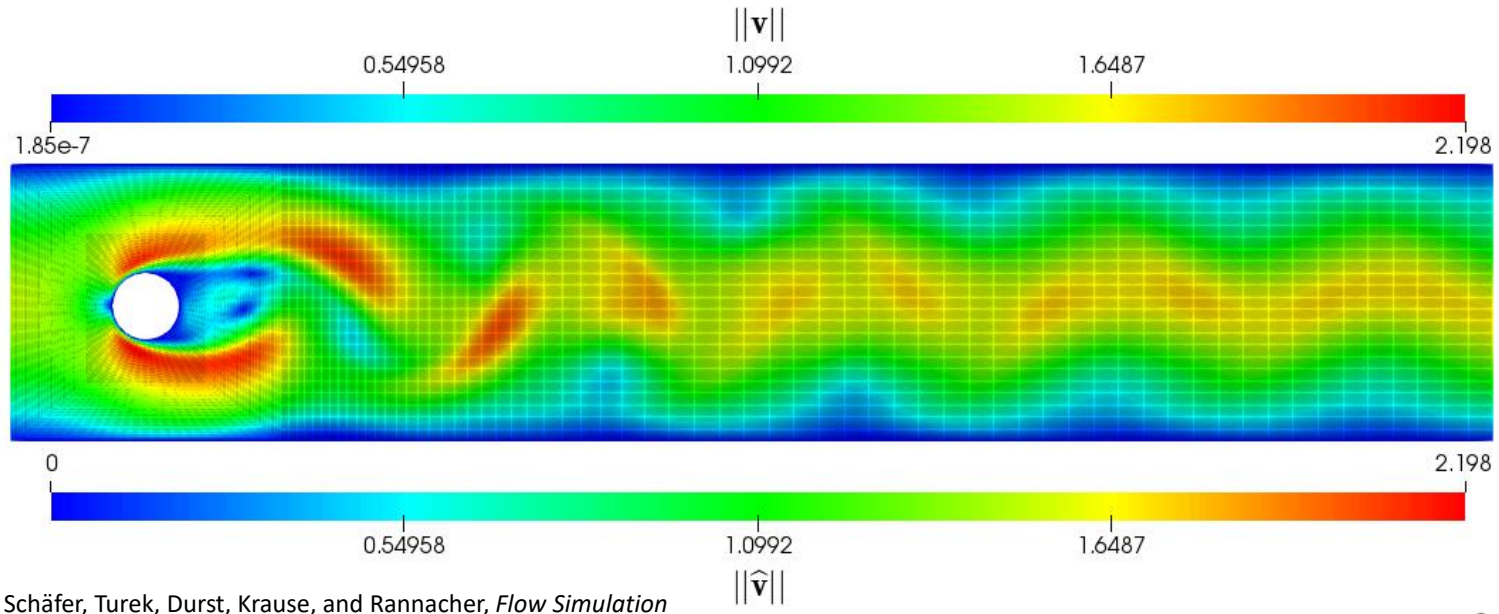
where $\mathbf{S}_{\mathrm{f}}$ *is defined as*

$$
\mathbf{S}_{\mathrm{f}} := 2\mu + \rho\|\mathbf{v}^{\mathrm{h}}\|.
$$

Benchmark

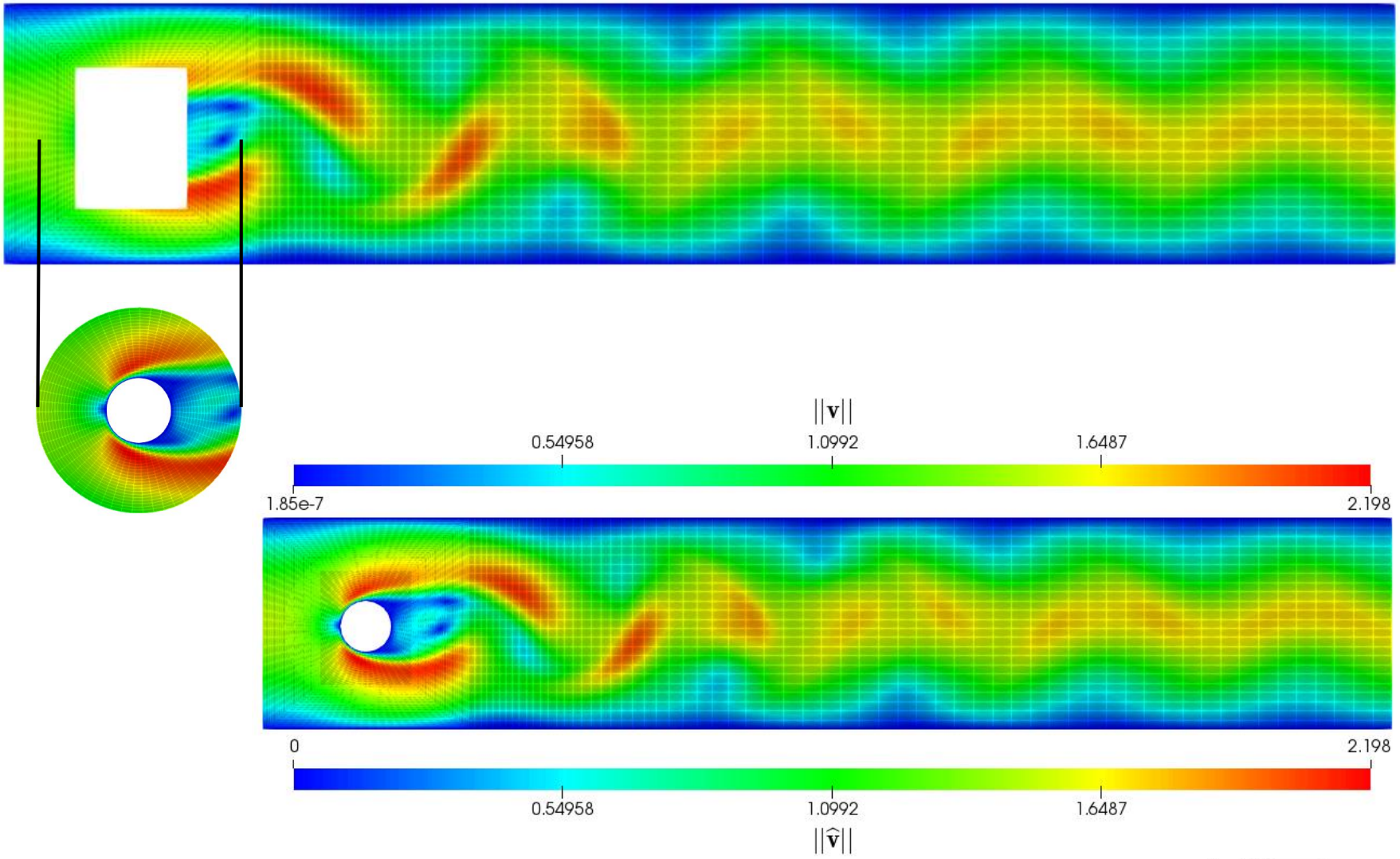


$\mu = 0.001\ Pa \cdot s$ $\rho = 1\ \frac{kg}{m^3}$ $\varepsilon = 4.54 \times 10^{-7}\ \frac{m \cdot s^2}{kg}$ $\Delta t = 0.0025\ s$



7. Schäfer, Turek, Durst, Krause, and Rannacher, *Flow Simulation with High-Performance Computers II*, 1996.

Results for Benchmark



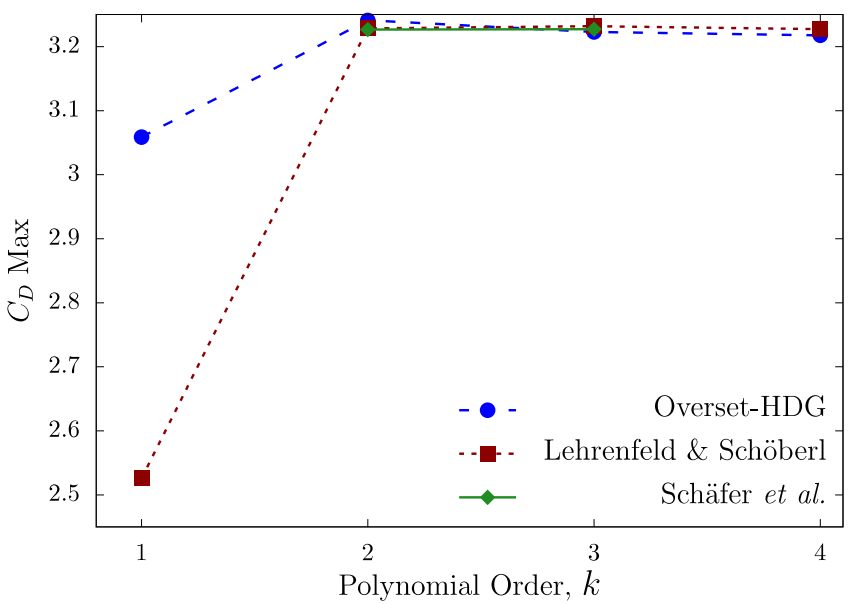
Results for Benchmark

	Order	# Global DoFs	C_D^{Max}	C_D^{Min}	C_L^{Max}	C_L^{Min}
OHDG	1	$\Omega^1 \cup \Omega^2$: 40,232 Ω^1: 29,140 Ω^2: 11,092	3.05870	3.05537	0.26514	-0.24157
Lehrenfeld & Schröberl	1	2,211	2.52594	2.47871	0.65728	-0.81672
OHDG	2	$\Omega^1 \cup \Omega^2$: 60,348 Ω^1: 43,710 Ω^2: 16,638	3.24096	3.16132	1.06156	-1.02484
Lehrenfeld & Schröberl	2	4,148	3.22841	3.16260	1.00571	-1.03894
OHDG	3	$\Omega^1 \cup \Omega^2$: 80,464 Ω^1: 58,280 Ω^2: 22,184	3.22271	3.14748	1.01633	-0.97931
Lehrenfeld & Schröberl	3	6,558	3.23184	3.16842	0.98822	-1.02427
OHDG	4	$\Omega^1 \cup \Omega^2$: 100,580 Ω^1: 72,850 Ω^2: 27,730	3.21757	3.14693	0.99833	-0.95634
Lehrenfeld & Schröberl	4	9,441	3.22714	3.16401	0.98431	-1.01906
Schäfer *et al.*	Q2/P1	167,232	3.22662	3.16351	0.98620	-1.02093
		667,264	3.22711	3.16246	0.98658	-1.02129

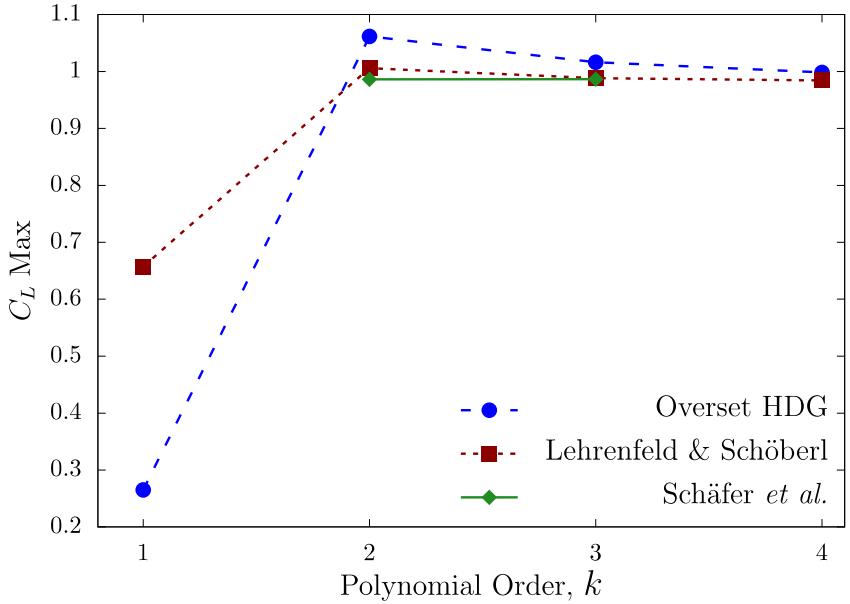
7. Schäfer, Turek, Durst, Krause, and Rannacher, *Flow Simulation with High-Performance Computers II*, 1996.
8. Lehrenfeld and Schöberl, *Computer Methods in Applied Mechanics and Engineering*, 2016.

Results for Benchmark

C_D Max



C_L Max



7. Schäfer, Turek, Durst, Krause, and Rannacher, *Flow Simulation with High-Performance Computers II*, 1996.
8. Lehrenfeld and Schöberl, *Computer Methods in Applied Mechanics and Engineering*, 2016.

ALE Navier-Stokes

Overset Isentropic ALE Navier-Stokes

Find $(\mathbf{L}_\mathrm{f}^\mathrm{h}, \mathbf{v}_\mathrm{f}^\mathrm{h}, p_\mathrm{f}^\mathrm{h}, \widehat{\mathbf{v}}_\mathrm{f}^\mathrm{h}) \in \mathscr{G}_\mathrm{h}^i \times \mathscr{Y}_\mathrm{h}^i \times \mathscr{P}_\mathrm{h}^i \times \mathscr{N}_\mathrm{h}^i$ *such that*

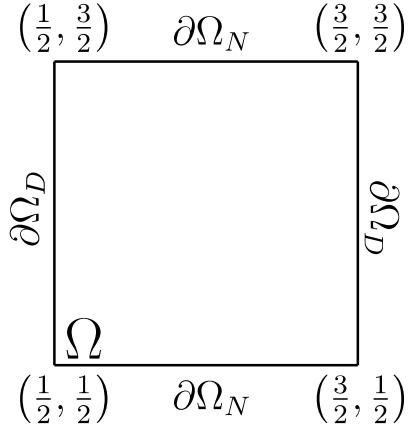
$$
\begin{aligned}
(\mathbf{G}, J_\mathrm{m}\mathbf{L}_\mathrm{f}^\mathrm{h})_{\mathcal{T}_\mathrm{h}^i} - (\mathbf{G}, J_\mathrm{m}\mathrm{Grad}\mathbf{v}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-1})_{\mathcal{T}_\mathrm{h}^i} + \left\langle \mathbf{G}, J_\mathrm{m}\left(\mathbf{v}_\mathrm{f}^\mathrm{h} - \widehat{\mathbf{v}}_\mathrm{f}^\mathrm{h}\right) \otimes \mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f}\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i} &= 0, \\
\left(\mathbf{y}, J_\mathrm{m}\rho_\mathrm{f}\frac{\partial \mathbf{v}_\mathrm{f}^\mathrm{h}}{\partial t}\right)_{\mathcal{T}_\mathrm{h}^i} + \left(\mathbf{y}, J_\mathrm{m}\rho_\mathrm{f}\mathbf{L}_\mathrm{f}^\mathrm{h}\left[\mathbf{v}_\mathrm{f}^\mathrm{h} - \mathbf{v}_\mathrm{m}\right]\right)_{\mathcal{T}_\mathrm{h}^i} + \left(\mathrm{Grad}\mathbf{y}, J_\mathrm{m}\mu_\mathrm{f}\mathbf{L}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\right)_{\mathcal{T}_\mathrm{h}^i} & \\
- \left(\mathrm{Grad}\mathbf{y}, J_\mathrm{m}p_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\right)_{\mathcal{T}_\mathrm{h}^i} + \left(\mathbf{y}, J_\mathrm{m}\rho_\mathrm{f}\left[\mathrm{Grad}\mathbf{v}_\mathrm{f}^\mathrm{h} : \mathbf{F}_\mathrm{m}^{-\intercal}\right]\mathbf{v}_\mathrm{f}^\mathrm{h}\right)_{\mathcal{T}_\mathrm{h}^i} & \\
+ \left\langle \mathbf{y}, J_\mathrm{m}\widehat{\mathbf{T}}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f}\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i} &= (\mathbf{y}, J_\mathrm{m}\mathbf{f})_{\mathcal{T}_\mathrm{h}^i}, \\
\left(q, J_\mathrm{m}\varepsilon\frac{\partial p_\mathrm{f}^\mathrm{h}}{\partial t}\right)_{\mathcal{T}_\mathrm{h}^i} - \left(q, J_\mathrm{m}\varepsilon p_\mathrm{f}^\mathrm{h}\left[\mathbf{F}_\mathrm{m}^{-\intercal}\mathrm{Grad}p_\mathrm{f}^\mathrm{h}\right]\cdot\mathbf{v}_\mathrm{m}\right)_{\mathcal{T}_\mathrm{h}^i} - \left(\mathrm{Grad}q, J_\mathrm{m}\mathbf{F}_\mathrm{m}^{-1}\mathbf{v}_\mathrm{f}^\mathrm{h}\right)_{\mathcal{T}_\mathrm{h}^i} + \left\langle q, J_\mathrm{m}\left[\widehat{\mathbf{v}}_\mathrm{f}^\mathrm{h}\cdot\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f}\right]\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i} &= (q, J_\mathrm{m}\gamma)_{\mathcal{T}_\mathrm{h}^i}, \\
\langle \boldsymbol{\eta}, J_\mathrm{m}\mathbf{v}_\mathrm{f}\rangle_{\Omega_{D_\mathrm{f}}^i} &= \langle \boldsymbol{\eta}, J_\mathrm{m}\mathbf{g}_D\rangle_{\Omega_{D_\mathrm{f}}^i}, \\
\left\langle \boldsymbol{\eta}, J_\mathrm{m}\widehat{\mathbf{T}}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f}\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i\setminus\Omega_{D_\mathrm{f}}^i} + \boxed{\left\langle \boldsymbol{\eta}, \breve{J}_\mathrm{m}\breve{\mathbf{T}}_\mathrm{f}^\mathrm{h}\breve{\mathbf{F}}_\mathrm{m}^{-\intercal}\breve{\mathbf{n}}_\mathrm{f}\right\rangle_{\partial\Omega_{O_\mathrm{f}}^i}} &= \langle \boldsymbol{\eta}, J_\mathrm{m}\mathbf{g}_N\rangle_{\Omega_{N_\mathrm{f}}^i},
\end{aligned}
$$

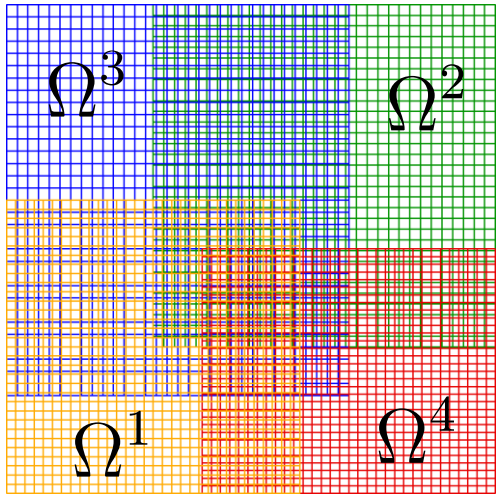
$\forall\,(\mathbf{G}, \mathbf{y}, q, \boldsymbol{\eta}) \in \mathscr{G}_\mathrm{h}^i \times \mathscr{Y}_\mathrm{h}^i \times \mathscr{P}_\mathrm{h}^i \times \mathscr{N}_\mathrm{h}^i$ *where*

$$
\begin{aligned}
\widehat{\mathbf{T}}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f} &= -\mu_\mathrm{f}\mathbf{L}_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f} + p_\mathrm{f}^\mathrm{h}\mathbf{F}_\mathrm{m}^{-\intercal}\mathbf{n}_\mathrm{f} + \mathbf{S}_\mathrm{f}\left(\mathbf{v}_\mathrm{f}^\mathrm{h} - \widehat{\mathbf{v}}_\mathrm{f}^\mathrm{h}\right), \\
\boxed{\breve{\mathbf{T}}_\mathrm{f}^\mathrm{h}\breve{\mathbf{F}}_\mathrm{m}^{-\intercal}\breve{\mathbf{n}}_\mathrm{f} = -\mu_\mathrm{f}\breve{\mathbf{L}}^\mathrm{h}\breve{\mathbf{F}}_\mathrm{m}^{-\intercal}\breve{\mathbf{n}}_\mathrm{f} + \breve{p}_\mathrm{f}^\mathrm{h}\breve{\mathbf{F}}_\mathrm{m}^{-\intercal}\breve{\mathbf{n}}_\mathrm{f} + \mathbf{S}_\mathrm{f}\left(\breve{\mathbf{v}}_\mathrm{f}^\mathrm{h} - \widehat{\mathbf{v}}_\mathrm{f}^\mathrm{h}\right)}.
\end{aligned}
$$

where $\mathbf{S}_\mathrm{f}$ *is defined as*

$$\mathbf{S}_\mathrm{f} := 2\mu_\mathrm{f} + \rho_\mathrm{f}\|\mathbf{v}_\mathrm{f}^\mathrm{h}\|.$$





Overset Mesh Motion

Find $(\mathbf{u}_\mathrm{m}^\mathrm{h}, \mathbf{F}_\mathrm{m}^\mathrm{h}, \widehat{\mathbf{u}}_\mathrm{m}^\mathrm{h}) \in \mathscr{U}_\mathrm{h}^i \times \mathscr{C}_\mathrm{h}^i \times \mathscr{M}_\mathrm{h}^i$ *such that*

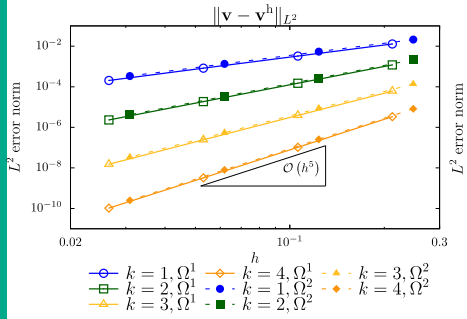
$$
\begin{aligned}
(\mathbf{C}, \mathbf{F}_\mathrm{m}^\mathrm{h} - \mathbf{I})_{\mathcal{T}_\mathrm{h}^i} - (\mathbf{C}, \mathrm{Grad}\mathbf{u}_\mathrm{m}^\mathrm{h})_{\mathcal{T}_\mathrm{h}^i} - \left\langle \mathbf{C}\mathbf{n}_\mathrm{m}, (\mathbf{u}_\mathrm{m}^\mathrm{h} - \widehat{\mathbf{u}}_\mathrm{m}^\mathrm{h})\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i} &= 0, 8 \\
\left(\mathrm{Grad}\mathbf{w}, \frac{1}{2}\mathbb{C}\left\{\left[\mathbf{F}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right] + \left[\mathbf{F}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right]^\intercal\right\}\right)_{\mathcal{T}_\mathrm{h}^i} & \\
- \left\langle \mathbf{w}, \widehat{\mathbf{P}}_\mathrm{m}^\mathrm{h}\mathbf{n}_\mathrm{m}\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i} &= (\mathbf{w}, \mathbf{b})_{\mathcal{T}_\mathrm{h}^i}, \\
\left\langle \omega, \widehat{\mathbf{u}}_\mathrm{m}^\mathrm{h}\right\rangle_{\Omega_{D_\mathrm{m}}^i} &= \langle \omega, \mathbf{t}_D\rangle_{\Omega_{D_\mathrm{m}}^i}, \\
\left\langle \omega, \widehat{\mathbf{P}}_\mathrm{m}^\mathrm{h}\mathbf{n}_\mathrm{m}\right\rangle_{\partial\mathcal{T}_\mathrm{h}^i\setminus\Omega_{D_\mathrm{m}}^i} + \boxed{\left\langle \omega, \breve{\mathbf{P}}_\mathrm{m}^\mathrm{h}\breve{\mathbf{n}}_\mathrm{m}\right\rangle_{\partial\Omega_{O_\mathrm{m}}^i}} &= \langle \omega, \mathbf{t}_N\rangle_{\Omega_{N_\mathrm{m}}^i},
\end{aligned}
$$

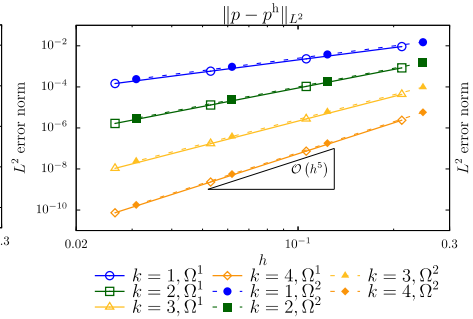
where

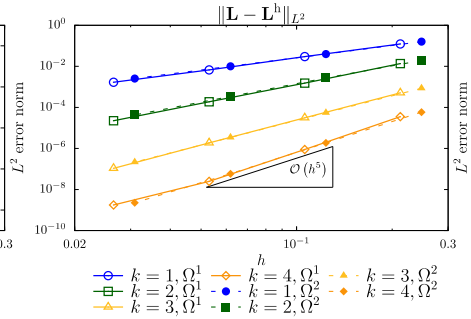
$$
\begin{aligned}
\widehat{\mathbf{P}}_\mathrm{m}^\mathrm{h}\mathbf{n}_\mathrm{m} &= \frac{1}{2}\mathbb{C}\left\{\left[\mathbf{F}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right] + \left[\mathbf{F}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right]^\intercal\right\}\mathbf{n}_\mathrm{m} - \mathbf{S}_\mathrm{m}\left(\mathbf{u}_\mathrm{m}^\mathrm{h} - \widehat{\mathbf{u}}_\mathrm{m}^\mathrm{h}\right), \\
\boxed{\breve{\mathbf{P}}_\mathrm{m}^\mathrm{h}\breve{\mathbf{n}}_\mathrm{m} = \frac{1}{2}\mathbb{C}\left\{\left[\breve{\mathbf{F}}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right] + \left[\breve{\mathbf{F}}_\mathrm{m}^\mathrm{h} - \mathbf{I}\right]^\intercal\right\}\breve{\mathbf{n}}_\mathrm{m} - \mathbf{S}_\mathrm{m}\left(\breve{\mathbf{u}}_\mathrm{m}^\mathrm{h} - \widehat{\mathbf{u}}_\mathrm{m}^\mathrm{h}\right)},
\end{aligned}
$$

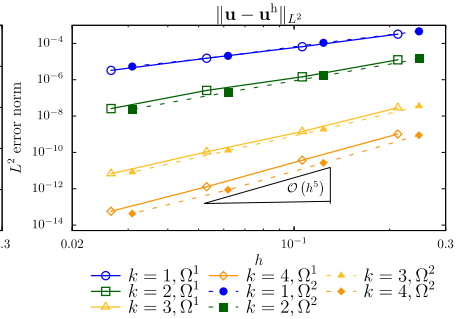
and $\mathbf{S}_\mathrm{m} = 50\mu_m\mathbf{I}$.

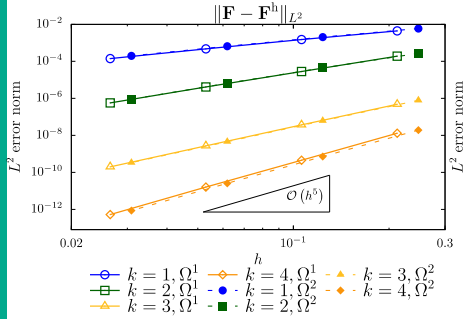
ALE Verification Results

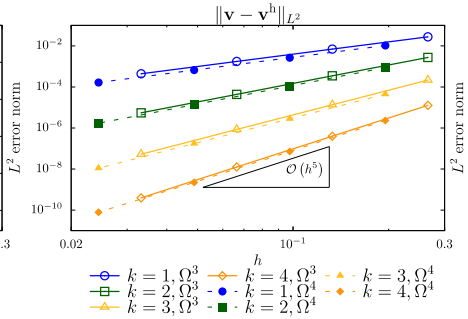


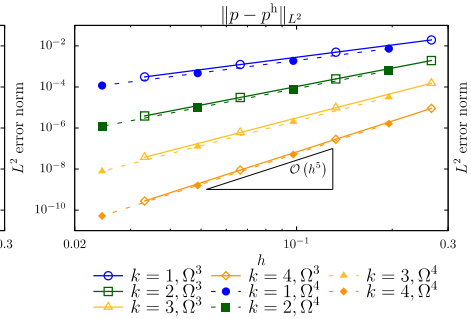


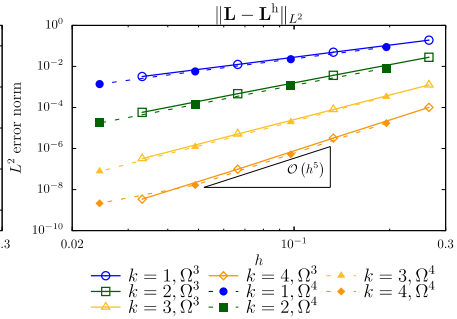


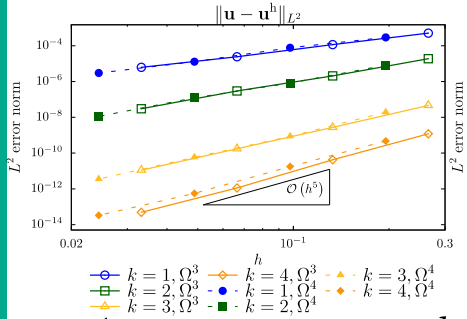


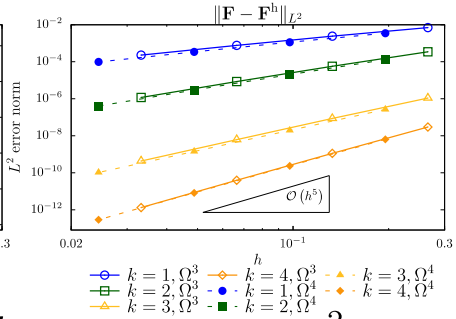






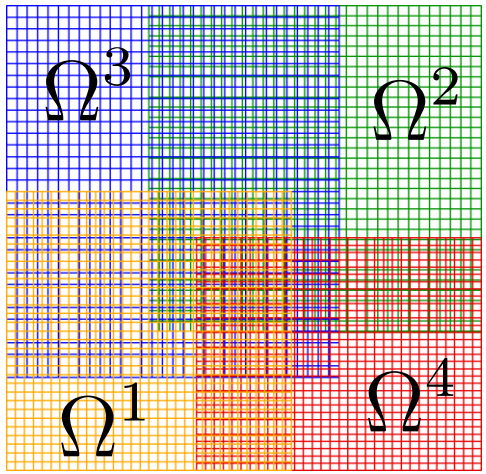




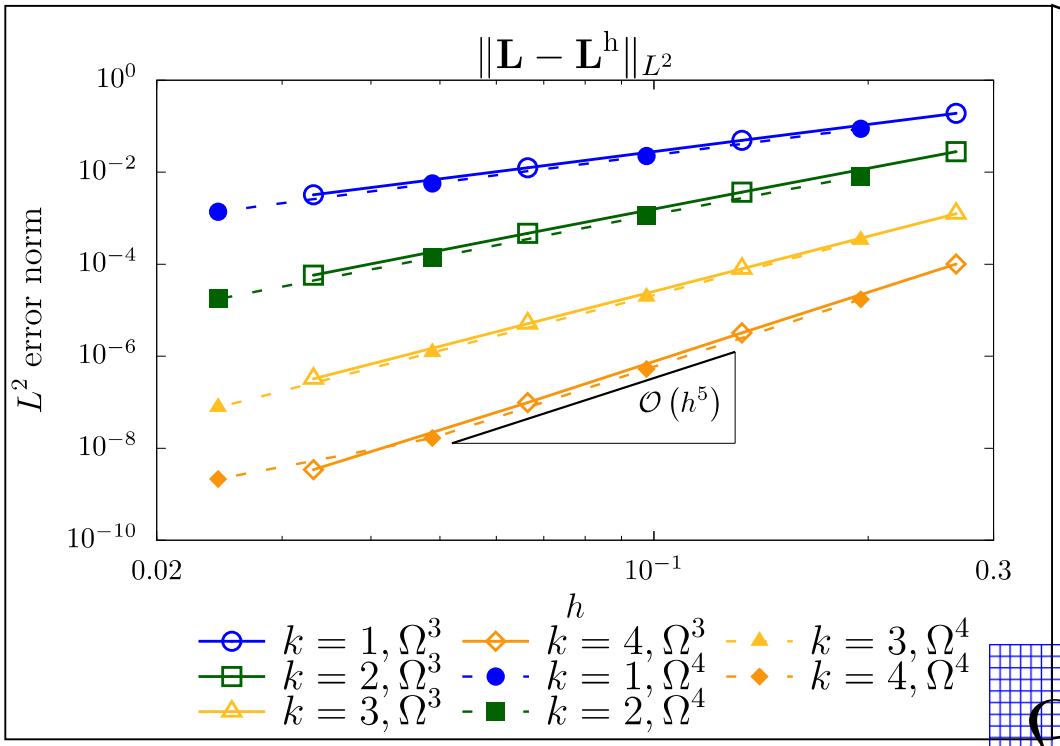


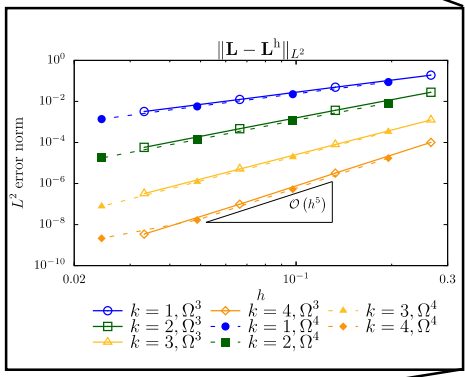
$$\mu = \frac{1}{2}\, Pa \cdot s \qquad \rho = 1\, \frac{kg}{m^3} \qquad \varepsilon = 1\, \frac{m \cdot s^2}{kg}$$

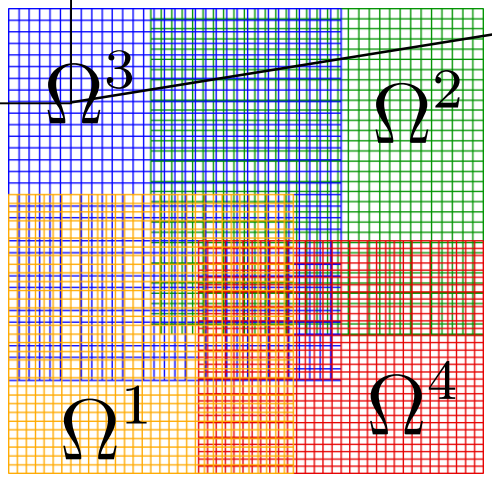
$$\Delta t = 10^{-6}\ s$$



ALE Verification Results







$$\mu = \frac{1}{2}\, Pa \cdot s \qquad \rho = 1\, \frac{kg}{m^3} \qquad \varepsilon = 1\, \frac{m \cdot s^2}{kg}$$

$$\Delta t = 10^{-6}\ s$$

Summary

Essential building-blocks for FSI problems have been verified, and optimal order convergence is maintained for an arbitrary amount of overlap.

Future Research

- Validation of overset-HDG ALE
- Formulation of fully coupled overset-HDG FSI
- Implementation will require
  - Dynamic meshes and Hole cutting
- Efficiency Improvements

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-- Questions? --